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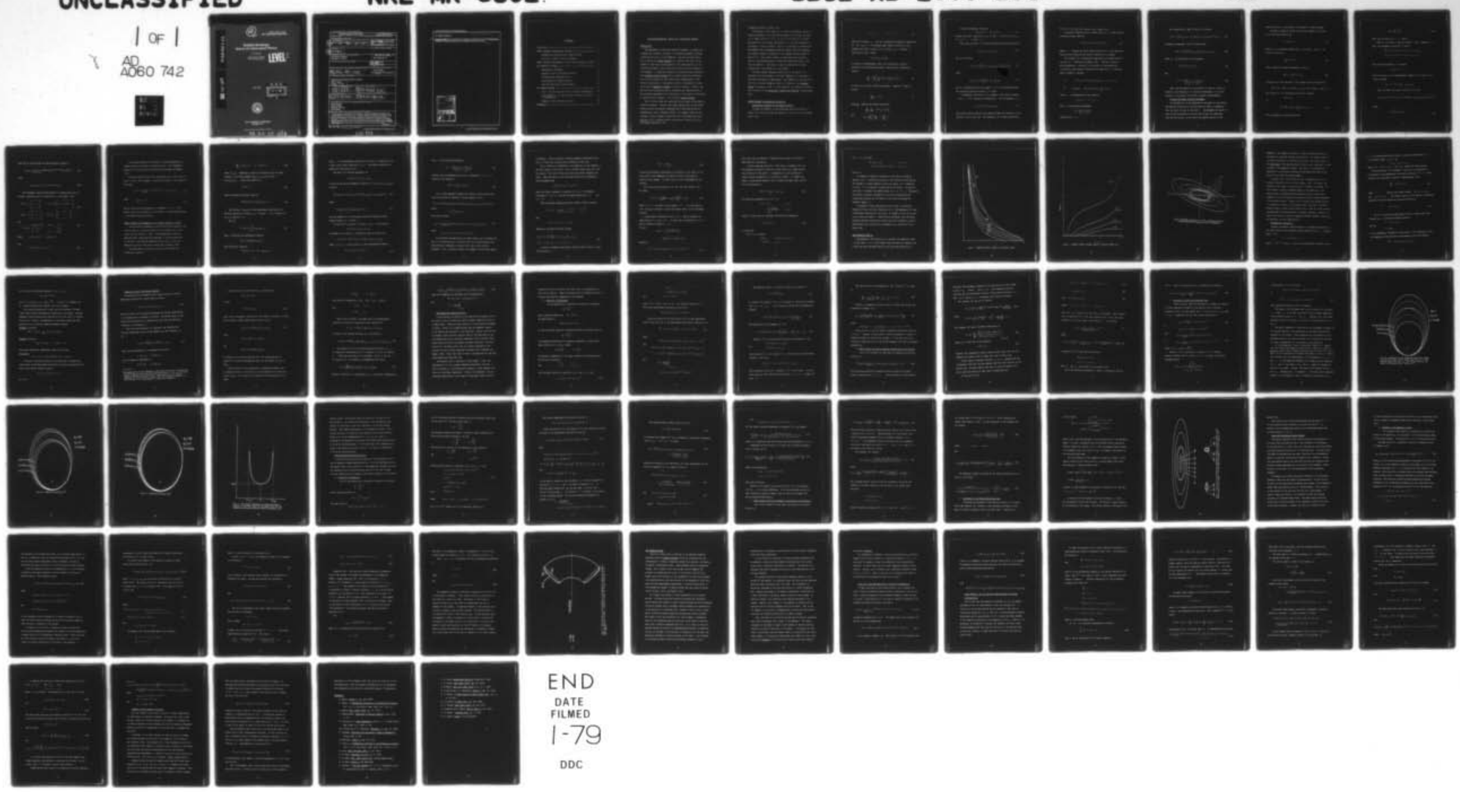

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NRL Memorandum Report 3802

Satellite Breakups: Survey of a Dynamical Theory

W. B. Heard

Systems Research Branch
Space Systems Division

LEVEL II

July 1978

NAVAL RESEARCH LABORATORY
Washington, D.C.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Memorandum Report 3802	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) SATELLITE BREAKUPS: Survey of a Dynamical Theory.		5. TYPE OF REPORT & PERIOD COVERED (9) Final report on an NRL problem.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) William B. Heard		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D. C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NRL Problem B01-10
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE (11) July 1978
		13. NUMBER OF PAGES 72
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (14) NRL-MR-3802		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(12) 73p.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

(18) SBIE (19) AD-E000 208

18. SUPPLEMENTARY NOTES

(16) RR01402 (17) RR0140241

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Satellite breakups
Particle dynamics
Phase-space continuum
Inverse problem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A general theory is developed for the dynamics of fragments resulting from the breakup of an artificial satellite or other celestial body. The particle ensemble is modeled as a continuum in phase space and the behaviour of the ensemble, as opposed to the behaviour of individual particles, is investigated. The solutions of four central problems are discussed. The problems are (i) the direct problem (prediction of the evolution of the ensemble from the initial conditions), (ii) the continuous source problem (direct problem for a point source active over an interval of time), (iii) the

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S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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as t goes to infinity

asymptotic problem (to find the state of the ensemble in the limit $t \to \infty$), and (iv) the inverse problem (deduction of the initial conditions from the ensemble structure at a later time).

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CONTENTS

SATELLITE BREAKUPS: SURVEY OF A DYNAMICAL THEORY

INTRODUCTION

The phenomenon of artificial satellite breakups, or sudden disintegrations, presents a plethora of interesting dynamical problems concerning the motion of the fragments. The basic problem, which will be called the direct problem, is to predict the motion of the ensemble of particles from knowledge of the characteristics of the disrupting mechanism, specifically the relative velocity imparted to the fragments. An important variation on the basic direct problem is the continuous source problem which considers particles dispersing over a finite interval of time rather than instantaneously. The eventual distribution of the particles after a long period of time, the so-called asymptotic problem, is also of interest. Finally, one may invert the problem and seek to determine the characteristics of the disrupting mechanism from observations of the particle distribution at some time after breakup -- the so-called inverse problem.

Each of the problems just mentioned has its place in the study of satellite breakups. However, their applicability does not end there because several astronomical phenomena may be mentioned which are represented by such a dynamical system. For example, there are the problems of meteor streams originating from a disintegrating comet (Jacchia, 1963), expanding stellar associations (Blaauw, 1952) and

Note: Manuscript submitted June 7, 1978.

fragmented asteroids (Wiesel, 1977).

The purpose of this report is to survey the dynamical theory of satellite breakups as it has been developed over the past two years or so (Heard, 1976; 1977 a,b). In view of the variety of possible applications of the theory, each aspect of it has been developed in as general a form as possible. This is not intended to distract the development from its primary motivation, namely satellite breakups, but rather to make it accessible to the largest possible sphere of application. This report consists of some previously published results, some generalizations of previously published results, and some new material. The goal is to give a complete and coherent presentation of the theory as it now stands.

The basic method underlying each facet of the theory is to represent the finite number of discrete fragments as a continuum of particles distributed in phase-space. This approach is borrowed from the field of stellar dynamics, where it is known as the continuum theory (Contopoulos, 1966). In this approach, all results are derived from solutions of the phase-space conservation equation (Chandrasekhar, 1960).

DIRECT PROBLEM (Instantaneous Disruption)

Mathematical Formulation and Formal Solution

Consider an ensemble of non-interacting particles moving in a common force field such that the equations of motion for an individual particle are

$$\dot{q} = X(q, p, t) \quad , \quad \dot{p} = Y(q, p, t) \quad , \tag{1}$$

where the n-vectors q, p are the coordinates and momenta, respectively. Let $f(q, p, t)$ be the phase space density function for the ensemble such that a volume $dq\,dp$ at point q, p contains dN particles at time t where

$$dN = f(q, p, t)\, dq\, dp \; .$$

According to Chandrasekhar (1960), the distribution function f satisfies the following first-order, linear partial differential equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left[\frac{\partial}{\partial q_i}(f X_i) + \frac{\partial}{\partial p_i}(f Y_i) \right] = 0 \tag{2}$$

if there is no source of particles present. Equation (2) may be written

$$\frac{Df}{Dt} = -f\Delta \tag{3}$$

where $\frac{D}{Dt}$ denotes the "Stokes derivative"

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + X \cdot \nabla_q + Y \cdot \nabla_p$$

and

$$\Delta = \sum_{i=1}^{n} \left[\frac{\partial X_i}{\partial q_i} + \frac{\partial Y_i}{\partial p_i} \right]$$

Introduce propagator functions

$$\underset{\sim}{Q}_\tau(\underset{\sim}{q},\underset{\sim}{p},t)\ ,\ \underset{\sim}{P}_\tau(\underset{\sim}{q},\underset{\sim}{p},t) \tag{4}$$

Defined such that a particle at $\underset{\sim}{q},\underset{\sim}{p}$ at time t will be at $\underset{\sim}{Q}_\tau(\underset{\sim}{q},\underset{\sim}{p},t)$ $\underset{\sim}{P}_\tau(\underset{\sim}{q},\underset{\sim}{p},t)$ at time $t+\tau$.

The formal solution of (3) which satisfies the initial condition

$$f(\underset{\sim}{q},\underset{\sim}{p},0) = F(\underset{\sim}{q},\underset{\sim}{p}) \tag{5}$$

may now be written

$$f(\underset{\sim}{q},\underset{\sim}{p},t) = F\left[\underset{\sim}{Q}_{-t}(\underset{\sim}{q},\underset{\sim}{p},t),\underset{\sim}{P}_{-t}(\underset{\sim}{q},\underset{\sim}{p},t)\right]\exp\{-\Gamma\} \tag{6}$$

where

$$\Gamma(\underset{\sim}{q},\underset{\sim}{p},t) = \int_0^t \Delta\left[\underset{\sim}{q}(t),\underset{\sim}{p}(t),t\right]dt, \tag{7}$$

and it is understood that the integral (7) is to be evaluated along the trajectory passing through $\underset{\sim}{q},\underset{\sim}{p}$ at time t.

A fundamental descriptor of the ensemble is the spatial density $\rho(\underset{\sim}{q},t)$. It is obtained by integrating f over all momenta, i.e.,

$$\rho(\underset{\sim}{q},t) = \int f(\underset{\sim}{q},\underset{\sim}{p},t)d\underset{\sim}{p} \quad . \tag{8}$$

The spatial density specifies the apparent shape and dispersion of the particle cloud at any time. The quadrature (8) is easily performed

formally if the particles emanate from a common point.

To describe dispersion from a common point, $\underset{\sim}{q}_*$, we may use the following initial condition:

$$F(\underset{\sim}{q}, \underset{\sim}{p}) = \delta(\underset{\sim}{q} - \underset{\sim}{q}_*)\, G(\underset{\sim}{p}) \quad , \tag{9}$$

where $\delta(\quad)$ denotes the Dirac delta-function and G is an arbitrary function which specifies the initial distribution of momenta.

The integral (8) is evaluated by applying two theorems from the calculus of δ- functions (Friedman, 1956). Theorem I relates a multi-dimensional δ-function to the product of one-dimensional δ-functions, and Theorem II specifies the behaviour of δ-functions under a change of variable:

I. $$\delta(\underset{\sim}{x} - \underset{\sim}{x}_o) = \delta(x_1 - x_{o,1})\,\delta(x_2 - x_{o,2}) \cdots \delta(x_n - x_{o,n})$$

II. $$\int \delta(\psi_1(\underset{\sim}{x})) \cdots \delta(\psi_n(\underset{\sim}{x}))\, F(\underset{\sim}{x})\, d\underset{\sim}{x} = \frac{1}{|J|}\, F(\underset{\sim}{x}_*) \quad ,$$

where $\underset{\sim}{x}_*$ is determined from the equations

$$\psi_i(\underset{\sim}{x}_*) = 0 \quad , \quad i = 1, \cdots, n \quad ,$$

and J is the Jacobian determinant

$$J = \frac{\partial(\psi_1, \cdots, \psi_n)}{\partial(x_1, \cdots, x_n)}$$

evaluated at $\underset{\sim}{x} = \underset{\sim}{x}_*$.

Now, substitute (9) and (6) into (8) to obtain

$$\rho(q,t) = \int \delta\left[Q_{-t}(q,p,t) - q_*\right] G\left[P_{-t}(q,p,t)\right] \exp\{-\Gamma\}\, dp \quad . \qquad (10)$$

According to Theorems I and II we then obtain

$$\rho(q,t) = \frac{1}{|J|} G\left[P_{-t}(q,p_*,t)\right] \exp\{-\Gamma(q,p_*,t)\} \quad , \qquad (11)$$

where p_* is the solution of the equation

$$Q_{-t}(q,p_*,t) = q_* \quad , \qquad (12)$$

and

$$J = \frac{\partial(Q_{-t,1}, \cdots, Q_{-t,n})}{\partial(p_1, \cdots, p_n)} \qquad (13)$$

Thus, the calculation of the evolution of spatial density is reduced to the evaluation of a Jacobian determinant (13) and the solution of an equation (12) which is usually transcendental.

Solution for Small Velocity Increments

In virtually all of the applications envisaged for the theory, the relative velocities of the particles are small in comparison with the total velocity of the parent. The mathematical import of this is that the equations of motion may usually be linearized. With this motivation, we now reduce the general results of the

previous section to the special form assumed for linear systems.

Consider a dynamical system for which the equations of motion of an individual particle are

$$\dot{\underset{\sim}{x}} = A \underset{\sim}{x} + \underset{\sim}{f} \tag{14}$$

where A is a non-singular matrix and $\underset{\sim}{f}$ is a vector. Let $\Psi(t)$ be a matrix solution of

$$\frac{d}{dt} \Psi = A \Psi \tag{15}$$

whose columns are linearly independent, and let

$$\Phi(t,t_0) = \Psi(t) \Psi^{-1}(t_0) \quad . \tag{16}$$

The matrix Φ is the matrizant of the system and has the properties

$$\Phi(t_0,t_0) = I \ , \ \Phi^{-1}(t_1,t_2) = \Phi(t_2,t_1) \ , \text{ and } \ \Phi(t_3,t_2)\Phi(t_2,t_1) = \Phi(t_3,t_1).$$

The solution of (14) satisfying the initial condition

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

is

$$\underset{\sim}{x} = \Phi(t,t_0)\underset{\sim}{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)\underset{\sim}{f}\, d\tau \quad . \tag{17}$$

Let us express Φ in partitioned form

$$\Phi = \left(\begin{array}{c|c} U & V \\ \hline W & Y \end{array}\right) \tag{18}$$

where each sub-block is a $n \times n$ matrix.

There are two cases of special interest. First, suppose $\underline{f} = 0$. Then, the propagator functions (4) become

$$\underline{Q}_\tau(q, p, t) = U(t+\tau, t)\, q + V(t+\tau, t)\, p \;,$$

$$\underline{P}_\tau(q, p, t) = W(t+\tau, t)\, q + Y(t+\tau, t)\, p \;.$$

The Jacobian determinant (13) becomes

$$J = \det V(0, t) \;,$$

and the solution of the transcendental equation (12), which is now linear, becomes

$$p_* = V^{-1}(0, t)\left[q_* - U(0, t)\, q \right]$$

Thus, we reduce the formal solution to the form

$$f(q, p, t) = G\left[W(0, t)\, q + Y(0, t)\, p\right] \delta\left[U(0, t)\, q + V(0, t)\, p - q_*\right] \exp\{-\gamma\} \tag{19}$$

where

$$\gamma = \int_0^t \mathrm{Tr}\, A(t)\, dt \;. \tag{20}$$

From this it follows that the spatial density reduces to

$$\rho(q,t) = \frac{1}{|\det V(0,t)|} G\left[M(0,t)q - Y(0,t)V^{-1}(0,t)q_{*}\right] \exp\{-\gamma\} \tag{21}$$

where

$$M(0,t) = W(0,t) - Y(0,t)V^{-1}(0,t)U(0,t) \tag{22}$$

The fundamental matrices appropriate for linearization about a circular, Keplerian orbit of mean motion n are (Heard, 1976)

$$U(t,\tau) = \begin{pmatrix} c & 0 & 0 \\ -2s & 1 & 0 \\ 0 & 0 & c \end{pmatrix} \qquad V(t,\tau) = \frac{1}{n}\begin{pmatrix} s & 2(1-c) & 0 \\ -2(1-c) & 4s-3(t-\tau) & 0 \\ 0 & 0 & s \end{pmatrix} \tag{23}$$

$$W(t,\tau) = \begin{pmatrix} -ns & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -ns \end{pmatrix} \qquad Y(t,\tau) = \begin{pmatrix} c & -2s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

and

$$M(t,\tau) = \begin{pmatrix} 3n(t-\tau)+4s & 2(1-c) & 0 \\ -2(1-c) & s & 0 \\ 0 & 0 & D/s \end{pmatrix} \tag{24}$$

where $s = \sin n(t-\tau)$, $c = \cos n(t-\tau)$

and

$$D = -3n(t-\tau) + 4(1-c) \quad . \tag{25}$$

In some applications of this case it is more appropriate to linearize about an elliptical, Keplerian orbit. The fundamental matrices for this case can be obtained from Tschauner and Hempel (1965).

The second special case worth elaborating upon is the case for which A and $\underline{f}$ are constants. Omitting the details, one obtains for this case

$$\rho(\underline{q}, t) = \frac{1}{|\det V|} \, G\left[M(\underline{q} + \underline{f}_1) + Y V^{-1}(\underline{q}_{*} + \underline{f}_1) - \underline{f}_2 \right] , \qquad (26)$$

where

$$\begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \end{pmatrix} = A^{-1} \underline{f} \; . \qquad (27)$$

and all matrices are evaluated at the arguments $(0 , t)$. This special case would be useful if one desired to include perturbations or thrusting through the vector $\underline{f}$.

DIRECT PROBLEM (Disintegration over a Finite Interval of Time)

In this section we generalize the direct problem treated in the previous section to allow the particles to disperse from a point source over an interval of time. This generalization would be important when considering the dispensing of orbiting dipoles as in the case of the Westford Needles (Morrow, et al., 1961) and when modelling dusty comet tails (Finson and Probstein, 1968).

The phase-space distribution function now satisfies the differential equation

$$\frac{Df}{Dt} + f\Delta = S \quad , \text{ or } \quad \mathcal{L}f = S \tag{28}$$

where $S(q,p,t)$ represents a source of S particles per unit time created in the volume element $dq\,dp$ at q, p and at time t.

Let $\phi(q,p,t|\tau)$ denote the solution of

$$\mathcal{L}\,\phi = 0 \quad , \tag{29}$$

which satisfies the initial condition

$$\phi(q,p,\tau|\tau) = G(p)\,\delta[q - q_s(\tau)] \quad . \tag{30}$$

The function $\phi(q,p,t|\tau)$ is the phase-space distribution for particles emitted at location $q_s(\tau)$ at time τ. It is defined for all q, p and for $t > \tau$.

Now let

$$f(q,p,t) = \int_{t_0}^{t} \phi(q,p,t|\tau)\,d\tau \quad . \tag{31}$$

Then f satisfies the differential equation

$$\mathcal{L}f = G(p)\,\delta[q - q_s(t)]$$

and the initial condition

$$f(q,p,t_0) = 0 \quad \text{for} \quad q \neq q(t_0) \; .$$

Thus, f is the phase-space distribution function for dispersion from a point source whose trajectory is $q_s(t)$ and whose distribution of momenta are described by $G(p)$.

The goal is to find an expression for

$$\rho(q,t) = \int f(q,p,t)\,dp \ .$$

To this end we use the propagator functions $Q_\tau(q,p,t), P_\tau(q,p,t)$, (4), to write

$$\phi(q,p,t|\tau) = G[P_{\tau-t}(q,p,t)]\,\delta[Q_{\tau-t}(q,p,t) - q_s(\tau)]\exp\{-\Gamma\} \tag{32}$$

where

$$\Gamma(q,p,t|\tau) = \int_\tau^t \Delta[q(t), p(t), t]\,dt \ , \tag{33}$$

and the integral is to be evaluated along that trajectory which passes through q, p at time t.

We are now in a position to obtain $\rho(q,p,t)$. Let us denote

$$\rho(q,t|\tau) = \int \phi(q,p,t|\tau)\,dp \ .$$

According to the results of the previous section we may write

$$\rho(q,t|\tau) = G[P_{\tau-t}(q,p_*,t)]\exp\{-\Gamma(q,p_*,t|\tau)\}\frac{1}{|J|} \ ,$$

where $p_*(q,t|\tau)$ is the solution of the transcendental equation

$$Q_{\tau-t}(q,p_*,t) = q_s(\tau) \ , \tag{34}$$

and J is the Jacobian determinant

$$J = \frac{\partial(Q_{\tau-t,1}, \ldots, Q_{\tau-t,n})}{\partial(p_1, \ldots, p_n)} \tag{35}$$

Finally, upon interchanging the order of integration, $\rho(q, t)$ is reduced to the quadrature

$$\rho(q, t) = \int_{t_0}^{t} \rho(q, t \mid \tau)\, d\tau \,. \tag{36}$$

For a linear dynamical system, the results of this and the previous section may be combined to reduce equation (32) to

$$\phi(q, p, t \mid \tau) = G\left[W(\tau,t)q + Y(\tau,t)p\right] \delta\left[U(\tau,t)q + V(\tau,t)p - q_s(\tau)\right] \exp\{-\gamma\}$$

where

$$\gamma = \int_{\tau}^{t} \mathrm{Tr}\, A(x)\, dx \,.$$

From this we obtain

$$\rho(q, t \mid \tau) = \frac{1}{|\det V(\tau,t)|} G\left[M(\tau,t)q - Y(\tau,t)V^{-1}(\tau,t)\, q_s(\tau)\right] \exp\{-\gamma\} \,, \tag{37}$$

where

$$M(\tau,t) = W(\tau,t) - Y(\tau,t)V^{-1}(\tau,t)U(\tau,t) \,.$$

To illustrate the application of these results, let us examine the case of a source moving on a circular orbit in a gravitational field whose epicyclic frequency is exactly twice that of the orbital frequency. Such a condition obtains, for example, in the inner regions

of galaxies. Given a suitably contrived momentum distribution function, a closed form solution can be obtained in this case.

Let us restrict our attention to the behaviour of the ensemble in the orbit plane of the source. Let us further choose units such that the radius of the orbit of the source and the orbital frequency are unity. Then the motion of the individual particles is determined from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2) - 2\xi p_2 + 2\xi^2$$

where the radial distance of a particle is $\varpi = 1 + \xi$, the angular coordinate is $\theta = \theta_0 + t + \eta$, and the conjugate mementa are $p_1 = \dot{\xi}$ and $p_2 = \dot{\eta} + 2\xi$.

From the general expressions given in Heard (1976) we obtain

$$M(t,\tau) = \frac{-1}{\sin(t-\tau)} \begin{pmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{pmatrix}$$

and

$$J = \sin^2(t-\tau) \quad , \quad \gamma = 0 \; .$$

Therefore, the partial density becomes

$$\rho(q,t|\tau) = \frac{1}{s^2}\, G\left[\frac{1}{s}(\xi s - \eta s)\, , \frac{1}{s}(\xi s + \eta c)\right] \tag{38}$$

with $s = \sin(t-\tau)$, $c = \cos(t-\tau)$.

A choice of momentum distribution function which allows a closed form solution is

$$G(p) = \frac{1}{1+p^2} \quad . \tag{39}$$

No particular physical significance is attached to this form of $G(p)$ other than it does emphasize the smaller ejection velocities as intuition would demand. Its main virtue is one of mathematical convenience.

After substituting equations (38) and (39) into equation (36) we obtain

$$\rho(\xi,\eta,t) = \frac{1}{q\sqrt{q^2+1}} \arctan\left[\frac{\sqrt{q^2+1}}{q} \tan t\right] \tag{40}$$

where $q^2 = \xi^2 + \eta^2$ and where we have taken $t_o = 0$. In using equation (40), care must be taken to choose the proper branch of the arctangent function.

A particularly revealing form for $\rho(q,t)$ may be obtained for large values of q , say $q \gg 1$. In this case, an expansion of $\rho(q,t)$ may be obtained in powers of $1/q^2$.

We have

$$\rho(q,t) = \int_0^t \frac{d\tau}{q^2 + \sin^2(t-\tau)}$$

$$\cong \frac{1}{q^2} \int_0^t \left(1 - \frac{1}{q^2}\sin^2 u\right) du$$

Therefore

$$\rho(q,t) \sim t/q^2 - (1/2q^4)(t - \tfrac{1}{2}\sin 2t) + O(1/q^6) \tag{41}$$

This shows that the density is asymptotically linear in time with a small periodic fluctuation.

A serious physical objection to the choice of equation (39) for the momentum distribution function is that there is no upper bound to the velocities of the ejecta. Consequently, as soon as ejection begins, particles appear at all locations in space. To remedy this defect, we may modify equation (39) to require an upper limit cut-off for the velocities as

$$G(p) = \frac{1}{1+p^2} H(p_0^2 - p^2) \quad . \tag{42}$$

The resulting expression for $\rho(q,t)$ is:

$$\begin{aligned} \rho(q,t) &= 0 & , \; q^2 > p_0^2 \\ &= \int_{\mathcal{S}} \frac{du}{q^2 + \sin^2 u} & , \; q^2 < p_0^2 \end{aligned}$$

where $\mathcal{S}$ is the union of intervals over which the inequality

$$p_0^2 > q^2 / \sin^2 u \tag{43}$$

is satisfied.

For $0 < t < \pi$ we have

$$\begin{aligned} \mathcal{S} &= \phi & , \; t < \delta = \arcsin(q/p_0) \\ &= [\delta, t] & , \; t > \delta \quad . \end{aligned}$$

For $\pi < t < 2\pi$ we have

$$\mathcal{S} = [\delta, \pi - \delta] \qquad , \quad t < \pi + \delta$$
$$= [\delta, \pi - \delta] \cup [\pi + \delta, t] \qquad , \quad t > \pi + \delta$$

and so on.

An example of numerical evaluation of the above is shown in Figures (1-2). A descriptive account of the evolution is as follows. The ensemble is always symmetric about the origin, i.e., bounded by a circle. It is originally concentrated at the origin. It grows in size until it reaches its maximum radius of p_0 at $t = \pi$. Thereafter the density increases everywhere in the circle as particles are continuously ejected but the radius of the circle enclosing the ensemble remains p_0.

An example of wider applicability is the case of continuous dispersion from a circular, Keplerian orbit. The quadrature (37) must be performed numerically in this case. An example of such an evaluation is shown in Figure 3. These results are based on an isotropic, Gaussian momentum distribution function and show isodensity contours when emission has occurred over one-quarter of a revolution of the parent body.

THE ASYMPTOTIC PROBLEM

The purpose of this section is to describe the asymptotic state (in the limit $t \to \infty$) of the ensemble when the particles emanate from a point and when the Hamiltonian for the individual particles is

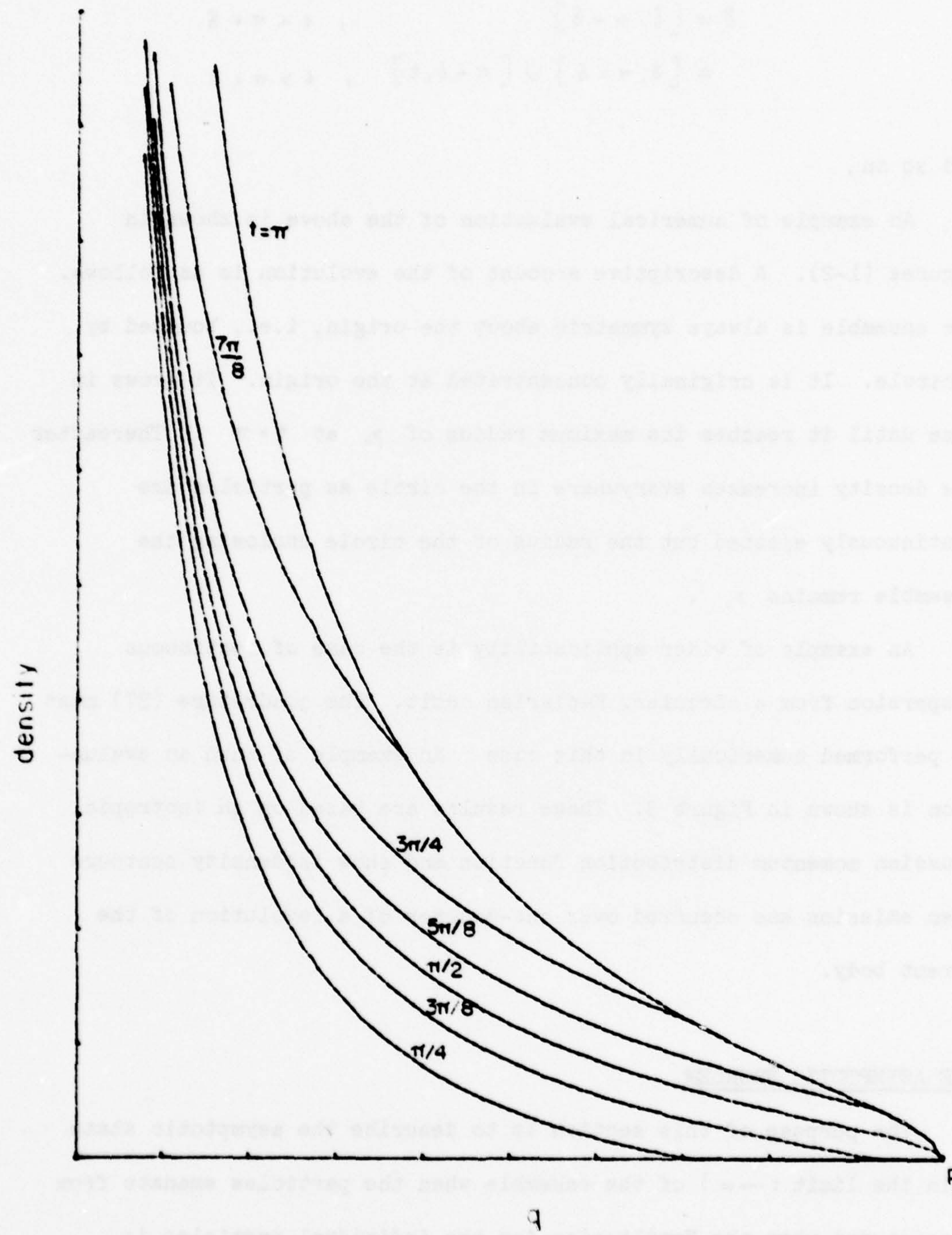

Fig. 1 - Spatial density versus q at various times

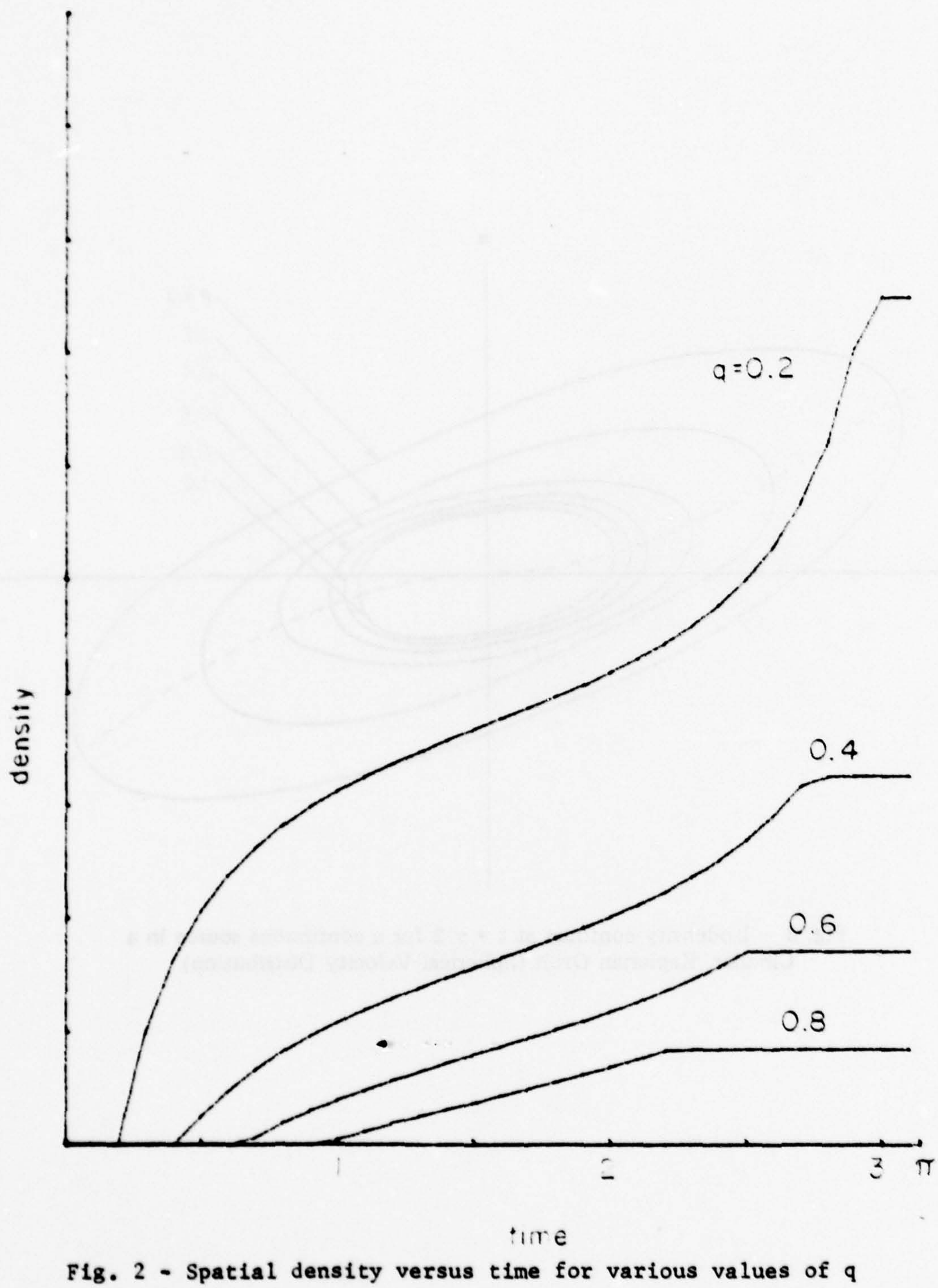

Fig. 2 - Spatial density versus time for various values of q

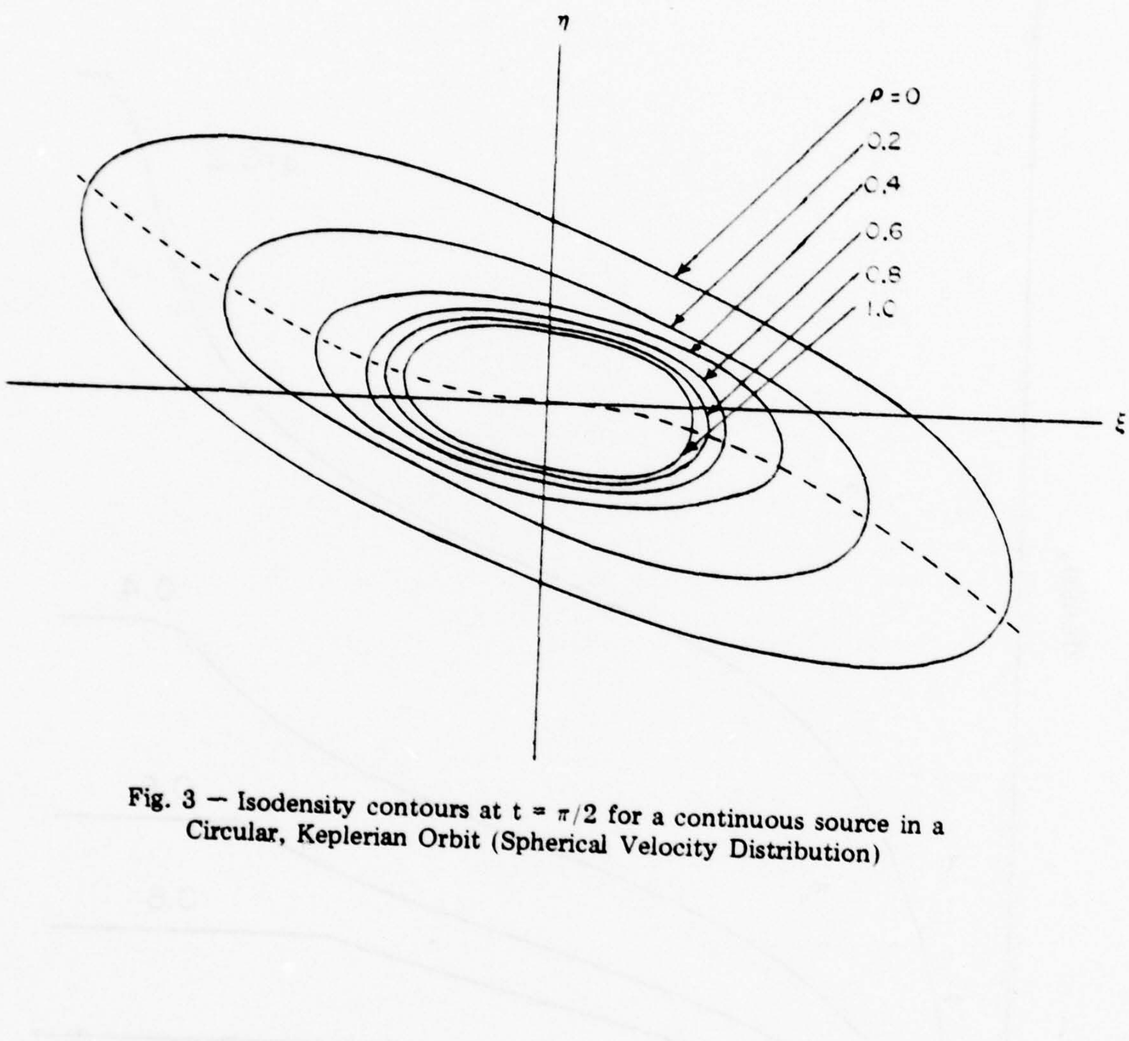

Fig. 3 — Isodensity contours at $t = \pi/2$ for a continuous source in a Circular, Keplerian Orbit (Spherical Velocity Distribution)

separable. The asymptotic behavior of the distribution function is determined by applying theorems developed by R. T. Prosser (1969) in his spectral analysis of central force motion. The theory leads to the <u>weak limit</u> of the spatial density function and, implicitly, to the boundary of the asymptotic domain. The theory is illustrated by application to the cases of elliptical Keplerian motion and to the case of motion about an oblate primary.

Kotsakis (1964) has studied the case of Keplerian motion for isotropic dispersion from a circular parent orbit. He obtains the boundary of the domain as the envelope of a family of ellipses. He also obtains a radially averaged density of the particles. The results from our dynamical theory generalize this to anisotropic dispersion from an elliptical orbit and provide the radial dependence of the spatial density. Wiesel (1976) considers the related problem of a statistical theory of the Kirkwood gaps. The basic difference between Wiesel's theory and the present one is that we invoke Prosser's theorems where he uses ergodicity on integral surfaces. Whereas our results are strictly limited to separable systems, Wiesel's theory has the potential of wider applicability.

<u>Mathematical Formulation</u>

Consider a dynamical system defined by a separable Hamiltonian, $\mathcal{H}$. The trajectories of this dynamical system define a flow, T_t ; on phase-space Π by

$$T_t : \underline{x}(\tau) \longrightarrow \underline{x}(t+\tau)$$

where $\underline{x} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix} \in \Pi$; and $\underline{q}, \underline{p}$ are the coordinates and conjugate momenta.

T_t is volume preserving and induces a unitary transformation, U_t , on the Hilbert space $\mathfrak{H} = L_2(\Pi)$ by

$$(U_t \varphi)(\underset{\sim}{z}) = \varphi(T_{-t}\underset{\sim}{z}) ,$$

where $\varphi \in \mathfrak{H}$. For $f, g \in \mathfrak{H}$; let (f, g) denote the inner product.

The distribution of the ensemble of particles in phase-space Π is described by the phase-space distribution function $F(\underset{\sim}{q}, \underset{\sim}{p}, t)$ which satisfies the conservation equation (Chandrasekhar, 1960)

$$\frac{\partial F}{\partial t} + \{F, \mathcal{H}\} = 0 , \qquad (44)$$

where $\{\ ,\ \}$ denotes the Poisson bracket. Let $\varphi_t(\underset{\sim}{z}) = F(\underset{\sim}{z}, t)$ and $\varphi_0(\underset{\sim}{z}) = F(\underset{\sim}{z}, 0)$. The formal solution of the partial differential equation (44) may be written in the above notation as

$$\varphi_t = U_t \varphi_0 . \qquad (45)$$

Let S be a canonical transformation from $\underset{\sim}{z}$ to some linear combination $\underset{\sim}{j}, \underset{\sim}{w}$ of action and angle variables,

$$S : \underset{\sim}{q}, \underset{\sim}{p} \longrightarrow \underset{\sim}{j}, \underset{\sim}{w}$$

and let

$$\underset{\sim}{\nu} = \mathcal{H}_{\underset{\sim}{j}}$$

be the fundamental frequencies of the motion. The composition of the flow mapping and the canonical transformation yields the mapping

$$ST_t : (\underset{\sim}{q}, \underset{\sim}{p}) \longrightarrow (\underset{\sim}{j}, \underset{\sim}{w} + t\underset{\sim}{\nu}) .$$

Let X be the infinitesimal generator of U_t, i.e.,

$$U_t = \exp\{-iXt\}$$

Let $\mathfrak{H}_0' = \{f \in \mathfrak{H} \cap \mathfrak{D}_X \mid Xf = 0\}$ and $\mathfrak{H}_0 = \overline{\mathfrak{H}_0'}$. If $\mathfrak{S} \subseteq \mathfrak{H}$ is a subspace, let $P_{\mathfrak{S}}$ denote the projection operator onto that subspace.

With these preliminaries, we may state the theorems of Prosser (1969) which form the mathematical foundation of our theory. Strictly speaking, Prosser proves these theorems only for the case of a central force field. However, an examination of the proofs shows that they generalize to an arbitrary separable dynamical system.

<u>Theorem 1</u> (Prosser):

For $f, g \in \mathfrak{H}$, $\lim\limits_{t \to \infty} (U_t f, g) = (P_{\mathfrak{H}_0} f, g)$

<u>Theorem 2</u> (Prosser):

$$P_{\mathfrak{H}_0} f = \int f \, d\underset{\sim}{w} \quad \text{if} \quad \underset{\sim}{\nu} \not\equiv 0 \quad \text{a.e.}$$

From these theorems we immediately obtain the corollary

<u>Corollary 1:</u>

$$\text{As } t \to \infty, \; U_t \varphi_0 \to \int \varphi_0 \, d\underset{\sim}{w} = \overline{\varphi} \quad (\text{weakly}).$$

Corollary 1 provides the basis for an algorithm to compute the (weak) limit of the phase-space distribution function and therefore the limit of the spatial density function,

$$\rho(\underset{\sim}{q}, t) = \int F(\underset{\sim}{q}, \underset{\sim}{p}, t) \, d\underset{\sim}{p},$$

as $t \to \infty$.

Asymptotic Form of the Spatial Density

For particles which disperse from a common point, the initial phase-space distribution function may be written

$$\varphi_0(\underset{\sim}{q},\underset{\sim}{p}) = \mathcal{F}(\underset{\sim}{p})\,\delta(\underset{\sim}{q}-\underset{\sim}{q}_0)\,.\,^* \tag{46}$$

The first factor is the function describing the initial distribution of the momenta and is assumed to be known. The second factor is the Dirac delta function and it describes the fact that the particles originate at coordinates $\underset{\sim}{q}_0$.

Since the transformation S is canonical, and therefore has Jacobian determinant unity, we may express φ_0 in the variables $\underset{\sim}{j},\underset{\sim}{w}$ as

$$\varphi_0(\underset{\sim}{j},\underset{\sim}{w}) = \mathcal{F}[\underset{\sim}{P}(\underset{\sim}{j},\underset{\sim}{w})]\,\delta[\underset{\sim}{Q}(\underset{\sim}{j},\underset{\sim}{w})-\underset{\sim}{q}_0] \tag{47}$$

where the transformation S is expressed in the form

$$\underset{\sim}{j} = \underset{\sim}{J}(\underset{\sim}{q},\underset{\sim}{p}) \quad , \quad \underset{\sim}{w} = \underset{\sim}{W}(\underset{\sim}{q},\underset{\sim}{p}) \quad ,$$

and its inverse is expressed as

$$\underset{\sim}{p} = \underset{\sim}{P}(\underset{\sim}{j},\underset{\sim}{w}) \quad , \quad \underset{\sim}{q} = \underset{\sim}{Q}(\underset{\sim}{j},\underset{\sim}{w}) \quad .$$

---

*The application of the theorems of the previous section to generalized functions has not been justified rigorously here. To do so, one would regard a generalized function as an equivalence class of regular sequences of good functions (Jones, 1966) and develop arguments for the validity of the interchange of limits and integrals.

Define the vector valued function $\underset{\sim}{w}_o$ implicitly by

$$\underset{\sim}{Q}(\underset{\sim}{j}, \underset{\sim}{w}_o) = \underset{\sim}{q}_o$$

so that

$$\underset{\sim}{w}_o = \underset{\sim}{w}_o(\underset{\sim}{j}, \underset{\sim}{q}_o) \ . \qquad (48)$$

Then, from a fundamental theorem about the change of variable for Dirac delta function (Jones, 1966) we may rewrite (47) as

$$\varphi_o(\underset{\sim}{j}, \underset{\sim}{w}) = \mathcal{G}\left[\underset{\sim}{R}(\underset{\sim}{j})\right] \delta(\underset{\sim}{w} - \underset{\sim}{w}_o)/J_1 \qquad (49)$$

where

$$\underset{\sim}{R}(\underset{\sim}{j}) = \underset{\sim}{P}\left[\underset{\sim}{j}, \underset{\sim}{w}_o(\underset{\sim}{j}, \underset{\sim}{q}_o)\right] \ ,$$

and

$$J_1(\underset{\sim}{j}) = \left| \det \underset{\sim}{Q}_{\underset{\sim}{w}}(\underset{\sim}{j}, \underset{\sim}{w}_o) \right| \ .$$

If equation (48) has more than one root, the right-hand-side of equation (49) must be evaluated at each root and summed over all of them.

We must allow for the possibility of degenerate systems, such as Keplerian motion, in which some of the frequencies are identically equal to zero. To do this, let us partition the variables into two parts

$$\underset{\sim}{w} = \begin{pmatrix} \underset{\sim}{w}' \\ \underset{\sim}{w}'' \end{pmatrix} \quad , \quad \underset{\sim}{j} = \begin{pmatrix} \underset{\sim}{j}' \\ \underset{\sim}{j}'' \end{pmatrix}$$

such that the frequencies $\underset{\sim}{\nu}' = \mathcal{H}_{\underset{\sim}{j}'}$ and $\underset{\sim}{\nu}'' = \mathcal{H}_{\underset{\sim}{j}''}$ satisfy

$$\underset{\sim}{\nu}' = 0 \quad (\text{a.e.})$$

and

$$\underset{\sim}{\nu}'' \neq 0 \quad (\text{a.e.}) \quad .$$

Then, from Corollary 1, the weak limit of the phase-space distribution function is expressed in new variables as

$$\bar{\varphi}(\underset{\sim}{j}, \underset{\sim}{w}') = \mathcal{G}[\underset{\sim}{R}(\underset{\sim}{j})] \, \delta[\underset{\sim}{w}' - \underset{\sim}{w}_o'(\underset{\sim}{j})] / J_1(\underset{\sim}{j}) \quad .$$

In terms of the original variables $\underset{\sim}{q}, \underset{\sim}{p}$ the result is

$$\bar{\varphi}(\underset{\sim}{q}, \underset{\sim}{p}) = \mathcal{G}\{\underset{\sim}{R}[\underset{\sim}{J}(\underset{\sim}{q}, \underset{\sim}{p})]\} \, \delta\{\underset{\sim}{W}'(\underset{\sim}{q}, \underset{\sim}{p}) - \underset{\sim}{w}_o'[\underset{\sim}{J}(\underset{\sim}{q}, \underset{\sim}{p})]\} / J_1[\underset{\sim}{J}(\underset{\sim}{q}, \underset{\sim}{p})] \tag{50}$$

The asymptotic form of the spatial density function, $\bar{\rho}(\underset{\sim}{q})$, is obtained by integrating the $\bar{\varphi}$ of equation (50) over all momenta $\underset{\sim}{p}$. There are two cases to be considered. First, if the set $\underset{\sim}{w}'$ is vacuous (i.e., the system is non-degenerate) we obtain

$$\bar{\rho}(\underset{\sim}{q}) = \int \left(\mathcal{G}\{\underset{\sim}{R}[\underset{\sim}{J}(\underset{\sim}{q}, \underset{\sim}{p})]\} / J_1[\underset{\sim}{J}(\underset{\sim}{q}, \underset{\sim}{p})] \right) d\underset{\sim}{p} \quad . \tag{51}$$

Second, if the set $\underset{\sim}{w}'$ is non-vacuous (i.e., the system is degenerate)

$$\bar{\rho}(\underset{\sim}{q}) = \int_{\Sigma} \left(\mathcal{F}\{\underset{\sim}{P}[\underset{\sim}{J}(\underset{\sim}{q},\underset{\sim}{p})]\} / J_1[\underset{\sim}{J}(\underset{\sim}{q},\underset{\sim}{p})]\, J_2[\underset{\sim}{J}(\underset{\sim}{q},\underset{\sim}{p})] \right) d\underset{\sim}{p} \qquad (52)$$

where the integration is performed over the hypersurface

$$\Sigma = \underset{\sim}{W}'(\underset{\sim}{q},\underset{\sim}{p}) - \underset{\sim}{w}_0[\underset{\sim}{J}(\underset{\sim}{q},\underset{\sim}{p})] = 0 \ ,$$

and

$$J_2 = \left| \nabla_{\underset{\sim}{p}} \Sigma \right| \Big|_{\Sigma = 0} \ .$$

Two-Dimensional Keplerian Motion

In this section the details of the algorithm are worked out for the case in which the particles execute bounded, Keplerian motion in a common plane. There are three reasons for considering this example in detail. First, it is simple enough that the algebraic details do not obscure the structure of the algorithm. Second, from the work of Kotsakis (1964) we have a partial check of the algorithm for the more specialized case of isotropic dispersion from a circular orbit. Third, when the velocity increments are small relative to the total velocity in the parent orbit, the out-of-plane component is decoupled from the in-plane components (Kotsakis 1964, Tschauner and Hempel, 1965). Thus, this case is really a prerequisite for the full, three-dimensional Keplerian case.

The analysis will be executed in three stages. First, the expression for $\bar{\rho}$ for a general momentum distribution function $\mathcal{F}$ will be reduced to a one dimensional quadrature which normally will have to be performed numerically. Second, the quadrature will be performed analytically for the case of anisotropic, small velocity

dispersion from an elliptical orbit when $\mathcal{F}$ may be represented as a Dirac delta function. Third, the results will be compared to those of Kotsakis and numerical examples will be presented.

A. Reduction to Quadrature

Let the generalized coordinates be the polar coordinates

$$q = \begin{pmatrix} r \\ \theta \end{pmatrix}$$

whose conjugate mementa are $p = \begin{pmatrix} \dot{r} \\ r^2 \dot{\theta} \end{pmatrix}$.

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2} (p_1^2 + p_2^2) - \mu / r \quad .$$

Let the coordinate system be oriented such that the initial point is

$$q_0 = \begin{pmatrix} r_0 \\ 0 \end{pmatrix} \quad .$$

The Delaunay variables are a convenient combination of action and angle variables so w and j become

$$w = \begin{pmatrix} \ell \\ g \end{pmatrix} \quad , \quad \nu = \begin{pmatrix} L \\ G \end{pmatrix} \quad .$$

The system is degenerate so the angle variables and frequencies are partitioned according to

and
$$\begin{aligned} w' &= g \quad &; \quad \nu' &= 0 \\ w'' &= \ell \quad &, \quad \nu'' &= \mu^2 L^{-3} \quad . \end{aligned}$$

The well-known relations connecting (q, p) and (w, j) are

$$L = \mu (2\mu/r - p_1^2 - p_2^2/r^2)^{-1/2} \tag{53}$$

$$G = p_2$$

$$r = G^2/\mu[1 + e\cos(\theta - g)]$$

and $$\theta = g + v(e, \ell)$$

where $e^2 = 1 - G^2/L^2$ and $v(e, \ell)$ is a periodic function of ℓ. The initial phase-space distribution function is

$$\varphi_0 = \mathcal{F}(p_1, p_2)\, \delta(r - r_0)\, \delta(\theta)/r_0 \quad .$$

Using the relations (53) and equation (50), we may immediately rewrite down the limit of the phase-space distribution function to be

$$\overline{\mathcal{F}}(r, \theta, p_1, p_2) = \mathcal{F}[\Psi_1(p_1, p_2), p_2]\, \delta(\eta)/|J|\, r_0 \qquad (54)$$

where

$$\Psi(p_1, p_2) = [p_1^2 + p_2^2(1/r^2 - 1/r_0^2) + 2\mu(1/r_0 - 1/r)]^{1/2}$$

and

$$\eta = \theta - \arccos\{[1 - (p_2^2/\mu)(2\mu/r - p_1^2 - p_2^2/r^2)]^{-1/2}(p_2^2/r\mu - 1)\}$$

$$+ \arccos\{[1 - (p_2^2/\mu)(2\mu/r - p_1^2 - p_2^2/r^2)]^{-1/2}(p_2^2/r_0\mu - 1)\} ,$$

and

$$J = \left. \frac{\partial(r, \theta)}{\partial(\ell, g)} \right|_{j, w_0}$$

$$= \mu(2\mu/r - p_1^2 - p_2^2/r^2)^{-3/2}\, \Psi_1(p_1, p_2) .$$

The remaining task is to obtain $\bar{\rho}$ which, as we recall, is

$$\bar{\rho} = \int \bar{\mathcal{G}} \, dp_1 \, dp_2 \,. \tag{55}$$

To evaluate the integral (55) it is convenient to transform coordinates from p_1, p_2 to η, p_2. In so doing, we obtain the one-dimensional integral

$$\bar{\rho} = \int \mathcal{G}\left\{ \Psi_1\left[p_1(0, p_2), p_2 \right], p_2 \right\} \left| \frac{\partial p_1}{\partial \eta} \right| \frac{dp_2}{|J| \, r_0} \tag{56}$$

The expression for the argument of $\mathcal{G}$ is

$$\chi(p_2) \equiv \Psi_1\left[p_1(0, p_2), p_2 \right] = \frac{\mu}{\sin\theta} \left[\frac{p_2}{\mu} \left(\frac{1}{r} - \frac{\cos\theta}{r_0} \right) - \frac{1}{p_2} (1 - \cos\theta) \right] \tag{57}$$

Equation (57) is most directly derived by eliminating g from

$$r = p_2^2/\mu \left[1 + e \cos(\theta - g) \right]$$

and

$$r_0 = p_2^2/\mu \, (1 + e \cos g)$$

then solving for p_1 as a function of p_2, and finally using the energy integral in the form

$$p_1^2 + p_2^2/r^2 - 2\mu/r = \chi^2 + p_2^2/r_0^2 - 2\mu/r_0 \,.$$

This derivation allows us to interpret $\chi(p_2)$ as the radial velocity which sends an orbit with initial conditions $r_0, 0; \chi, p_2$ through the point r, θ.

The third factor in the integrand of (56) involves $\partial p_1/\partial\eta$ which is

$$\frac{\partial p_1}{\partial \eta} = \frac{\mu}{\sin^2\theta}\left[\frac{p_2}{\mu}\left(\frac{\cos\theta}{r_0} - \frac{1}{r}\right) + \frac{1}{p_2}\,(1-\cos\theta)\right] \qquad (58)$$

Finally, we assemble the above results to obtain the desired one-dimensional integral for $\bar{\rho}$:

$$\bar{\rho} = \int \mathcal{F}\left[\frac{(\beta_0 c-\beta)\mu}{s\,p_2},\, p_2\right] \frac{\mu^2}{s^4 p_2^3} \left|s^2-\beta^2-\beta_0^2+2\beta\beta_0 c\right|^{3/2} \frac{dp_2}{r_0} \qquad (59)$$

where

$$\beta = p_2^2/\mu r - 1 \quad , \quad \beta_0 = p_2^2/\mu r_0 - 1 \ , \quad c = \cos\theta \quad \text{and} \quad s = \sin\theta \ .$$

This solution is valid for general momentum distribution functions $\mathcal{F}(p_1, p_2)$. However, in most cases it would be necessary to resort to numerical methods to evaluate the integral. In the next section we consider an important case for which the integrand (59) can be evaluated analytically.

B. <u>Small Initial Velocity Increments Concentrated on an Ellipse</u>

Let us now consider the case where the momentum distribution function is

$$\mathcal{F}(p_1,p_2) = \frac{1}{\pi\sigma_1\sigma_2 r_0 v_0^2}\ \delta\left[(p_1-p_{10})^2/\sigma_1^2 + (p_2-p_0)^2/r_0^2\sigma_2^2 - v_0^2\right] \qquad (60)$$

This describes anisotropic dispersion from the parent orbit whose initial conditions are $r_0, 0; p_{10}, p_0$. The anisotropy is of the special

form that the incremental momenta of the particles lie on an ellipse in the p_1, p_2 -plane. Were $\sigma_1 = \sigma_2$, the dispersion would be isotropic and the incremental velocity of each particle would be $\sigma_1 v_0$. Were $p_{10} = 0$ and $p_0^2/\mu = r_0$, the parent orbit would be circular.

The integral (59) may now be written

$$\bar{\rho} = \int \delta[\xi(p_2)] \frac{\mu^2}{s^4 p_2^3} \left| s^2 - \beta^2 - \beta_0^2 + 2\beta\beta_0 c \right|^{3/2} \frac{dp_2}{\pi r_0^2 v_0^2 \sigma_1 \sigma_2} \tag{61}$$

where

$$\xi(p_2) = \frac{1}{\sigma_1^2 s^2} \left[\frac{\beta_0 c - \beta}{p_2} - p_{10} s \right]^2 + \frac{(p_2 - p_0)^2}{r_0^2 \sigma_2^2} - v_0^2 \quad .$$

The integral (61) may be evaluated immediately as

$$\bar{\rho}(r, \theta) = \sum_{p_2^*} \left. \frac{\frac{\mu^2}{s^4 p_2^3} \left| s^2 - \beta^2 - \beta_0^2 + 2\beta\beta_0 c \right|^{3/2}}{(d\xi/dp_2)\, \pi r_0^2 v_0^2 \sigma_1 \sigma_2} \right|_{p_2 = p_2^*} \tag{62}$$

where p_2^* is the root of the equation

$$\xi(p_2^*) = 0 \quad . \tag{63}$$

Equation (63) represents a quartic which has two real roots, and the summation in equation (62) is taken over both of these roots.

In most cases of physical interest $r_0 v_0 / p_0 \ll 1$, meaning that the incremental velocities are much smaller than the total velocity of the parent body. We shall exploit this fact to solve the quartic (63) and to give the results in their most illuminating form.

To this end, we let

$$\bar{r}(\theta) = 1/\left[1 + (p_0^2/\mu r_0 - 1)c - (p_0 p_{10}/\mu)s\right] \quad , \tag{64}$$

$$r = (p_0^2/\mu)(\bar{r} + x) \quad , \tag{65}$$

and
$$p_* = p_0\left[1 + \epsilon(x)\right] \quad , \tag{66}$$

where $|x| \; , |\epsilon| \ll \bar{r} \sim 1 \quad , \epsilon(0) = 0 \; .$

The curve $r(\theta) = (p_0^2/\mu)\bar{r}(\theta)$ is the orbit of the parent. Upon substitution of equations (65) - (66) into equations (62) - (63), and after retention of only the lowest order terms in x , one obtains

$$\bar{\rho}(x,\theta) = \frac{\bar{r}^2(1-e_0^2)^{3/2}}{\pi\sigma_1\sigma_2 p_0^2 (v_0/v_c)^2}\,\Delta^{-1/2} \quad , \tag{67}$$

where

$$\Delta(x,\theta) = \left(\frac{r_0 v_0}{p_0}\right)^2 \bar{r}^{-4}\left\{\left(\frac{p_0^2}{\mu r_0}\right)^2 \frac{s^2}{\sigma_2^2} + \frac{\left[2(1-c) - (p_0 p_{10}/\mu)s\right]^2}{\sigma_1^2}\right\} - \frac{x^2}{\sigma_1^2\sigma_2^2} \tag{68}$$

In equation (67) we have used the relations

$$-p_{10}p_0/\mu = e_0 \sin g_0 \quad ,$$

$$p_0^2/r_0\mu = 1 + e_0 \cos g_0 \quad ,$$

where e_0 and g_0 are elements of the parent orbit.

The curve bounding the asymptotic domain is obtained by setting

$\Delta = 0$. Thus, the bounding curve is defined by the equation

$$x_{\pm}(\theta) = \pm \bar{r}^2 \frac{r_0 v_0}{p_0} \left\{ \sigma_1^2 s^2 \left(\frac{p_0^2}{\mu r_0} \right)^2 + \sigma_2^2 \left[2(1-c) - p_0 p_{10} s/\mu \right] \right\}^{1/2} \tag{69}$$

C. Discussion of the Two-Dimensional Case

First we should take the opportunity to compare our results to those obtained by Kotsakis in the special case of isotropic dispersion from a circular parent orbit. If we let $p_0^2/\mu = r_0$, $p_{10} = 0$, and $\sigma_1 = \sigma_2 = 1$; equations (69) and (68) become, respectively,

$$x_{\pm} = \pm \sqrt{r_0/\mu}\; v_0 \left[s^2 + 4(1-c)^2 \right]^{1/2} \tag{70}$$

or

$$x_{\pm} = \pm 2\alpha \sin\tfrac{1}{2}\theta \left(1 + 3\sin^2\tfrac{1}{2}\theta \right)^{1/2}$$

and

$$\bar{\rho} = \frac{1}{\pi p_0^2 \alpha^2} \left\{ \alpha^2 \left[s^2 + 4(1-c)^2 \right] - x^2 \right\}^{1/2}$$

or

$$\bar{\rho} = \frac{1}{\pi p_0^2 \alpha^2} \left\{ 4\alpha^2 \sin^2\tfrac{1}{2}\theta \left(1 + 3\sin^2\tfrac{1}{2}\theta \right) - x^2 \right\}^{-1/2} \tag{71}$$

where $\alpha = v_0/v_c$ and $v_c = \sqrt{\mu/r_0}$.

Equation (70) is equivalent to equation (14) of Kotsakis. However, Kotsakis does not derive $\bar{\rho}(x,\theta)$ but rather the radially averaged density

$$\bar{\bar{\rho}}(\theta) = \frac{\displaystyle\int_{x_-}^{x_+} \bar{\rho}\, dx}{\displaystyle 2x_+ \int_0^{2\pi}\!\int_{x_-}^{x_+} \bar{\rho}\, r\, dr\, d\theta}$$

Using equation (71) we calculate

$$\bar{\bar{\rho}} = 1/(2\pi r_0^2)(4\alpha) \sin\tfrac{1}{2}\theta\,(1 + 3\sin^2\tfrac{1}{2}\theta)^{1/2}$$

or

$$\bar{\bar{\rho}} = 1/4r_0T_0 \sin\tfrac{1}{2}\theta\,(1 + 3\sin^2\tfrac{1}{2}\theta)^{1/2} \tag{72}$$

which agrees with equation (21') of Kotsakis.

The shape of the bounding curve is quite sensitive to the value of g_0 when e_0 is not small and there is also a marked dependence on the relative values of σ_1 and σ_2. Figures (4a-c) show these bounding curves for the case $e_0 = 0.7$ and for various values of g_0, σ_1 and σ_2.

The radial dependence of the density is illustrated in Figure (5). The salient feature is the singularity at the inner and outer extremeties of the domain. These singularities are artifacts of the singular nature of the function chosen for $\mathcal{F}$ and would be smoothed out upon a superposition of a continuum of such functions.

It is interesting to invert the problem. That is, to suppose that one observes an asymptotic distribution of particles, known <u>a priori</u> to result from a momentum distribution of the form (60), and to ask if we can determine this momentum distribution function. Clearly, this involves only the determination of the parameters $p_{10}, p_0, r_0, \theta_0, \sigma_1, \sigma_2$, and v_0. If one knows the boundary of the domain one can determine $\bar{r}(\theta)$, r_0 and θ_0 and hence r_0, θ_0, p_{10} and p_0 simply by tracing the center of the domain. Further, the shape of the boundary yields $v_0\sigma_1$ and $v_0\sigma_2$ unambiguously. To separate v_0 from these latter paramters, however, it is necessary to know, in addition, the density at an

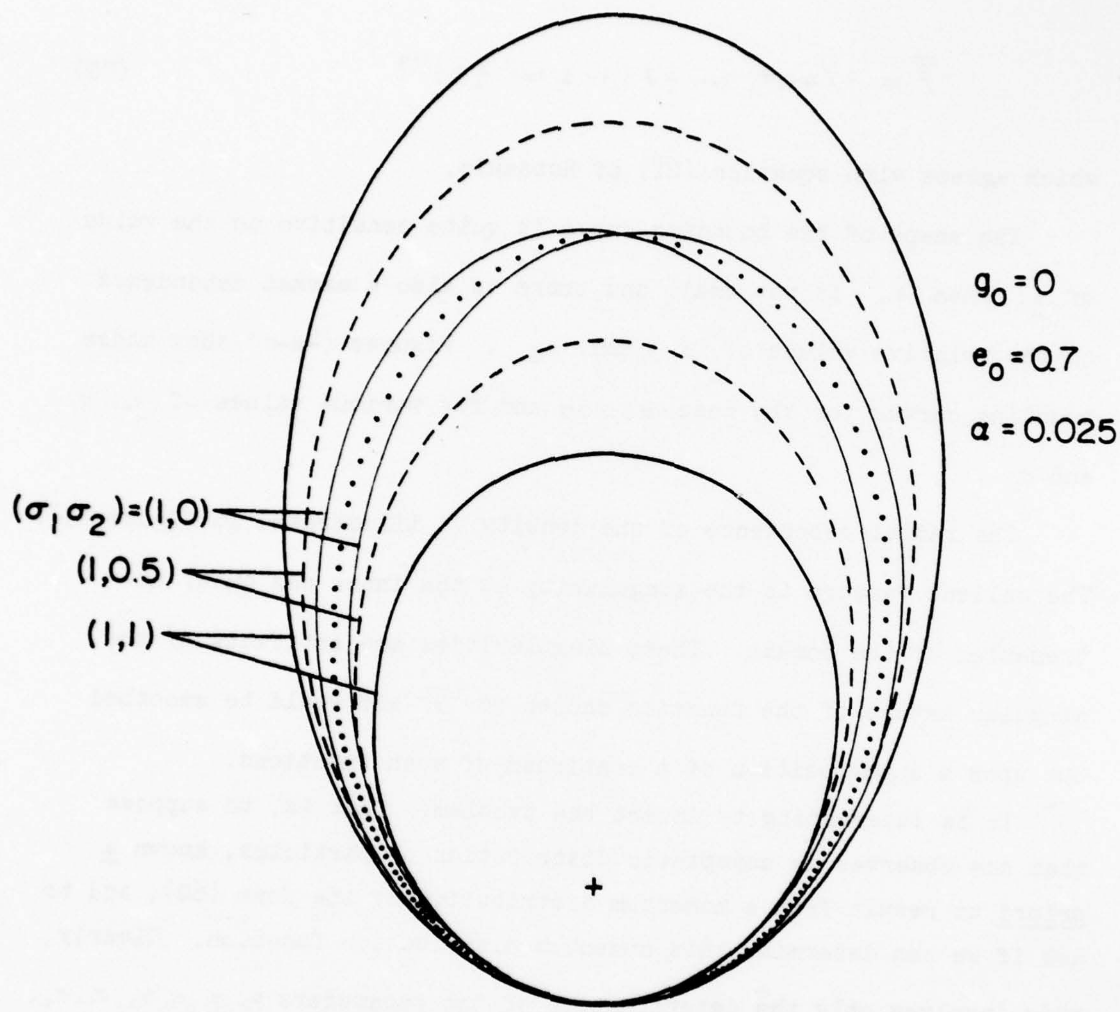

Fig. 4a - Boundary for the Keplerian asymptotic domain in the orbit plane of the parent body for $e_o = 0.7$ and various values of σ_1 and σ_2. The dotted curve is the orbit of the parent body. Here $g_o = 0^o$.

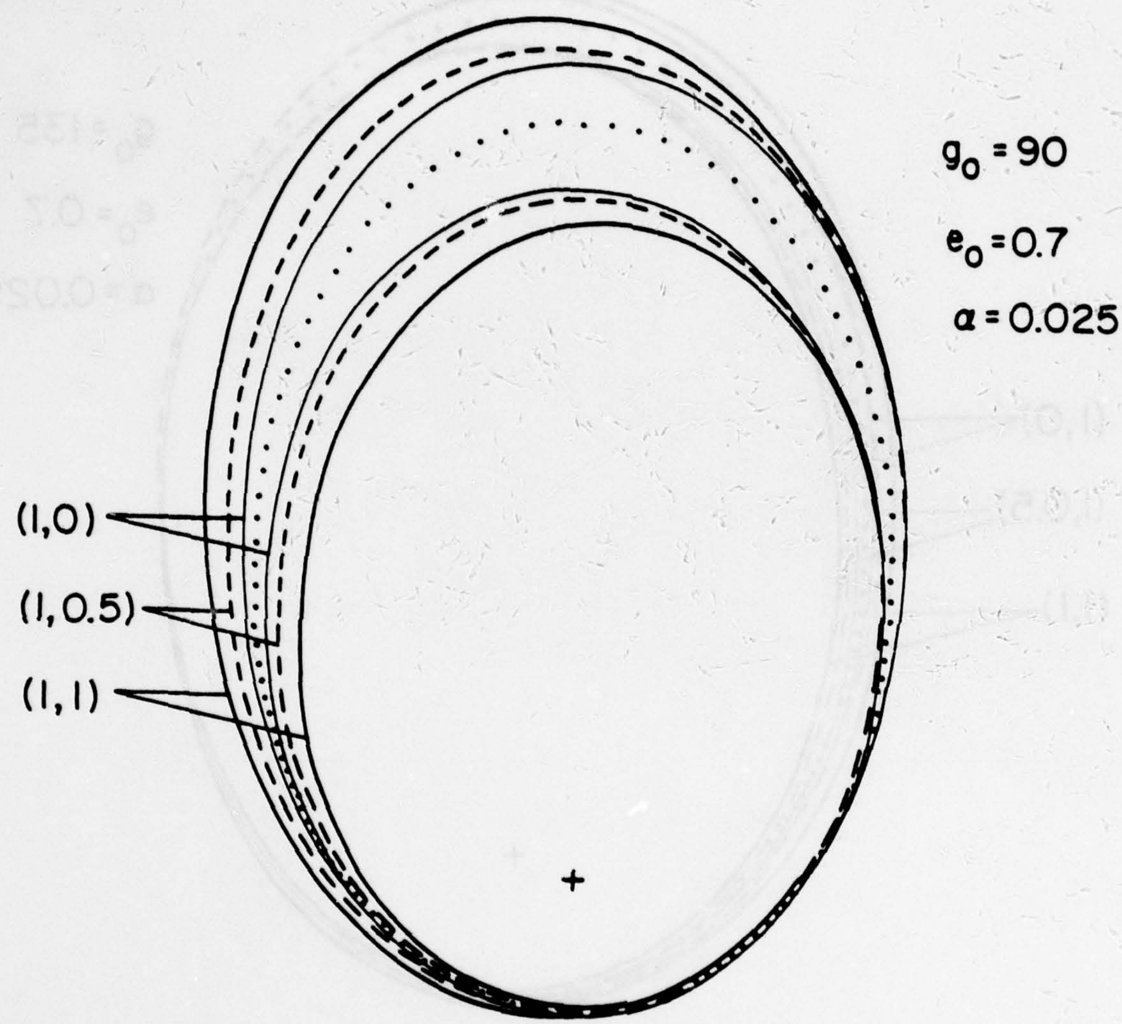

Fig. 4b - Same as 4a but $g_o = 90^\circ$

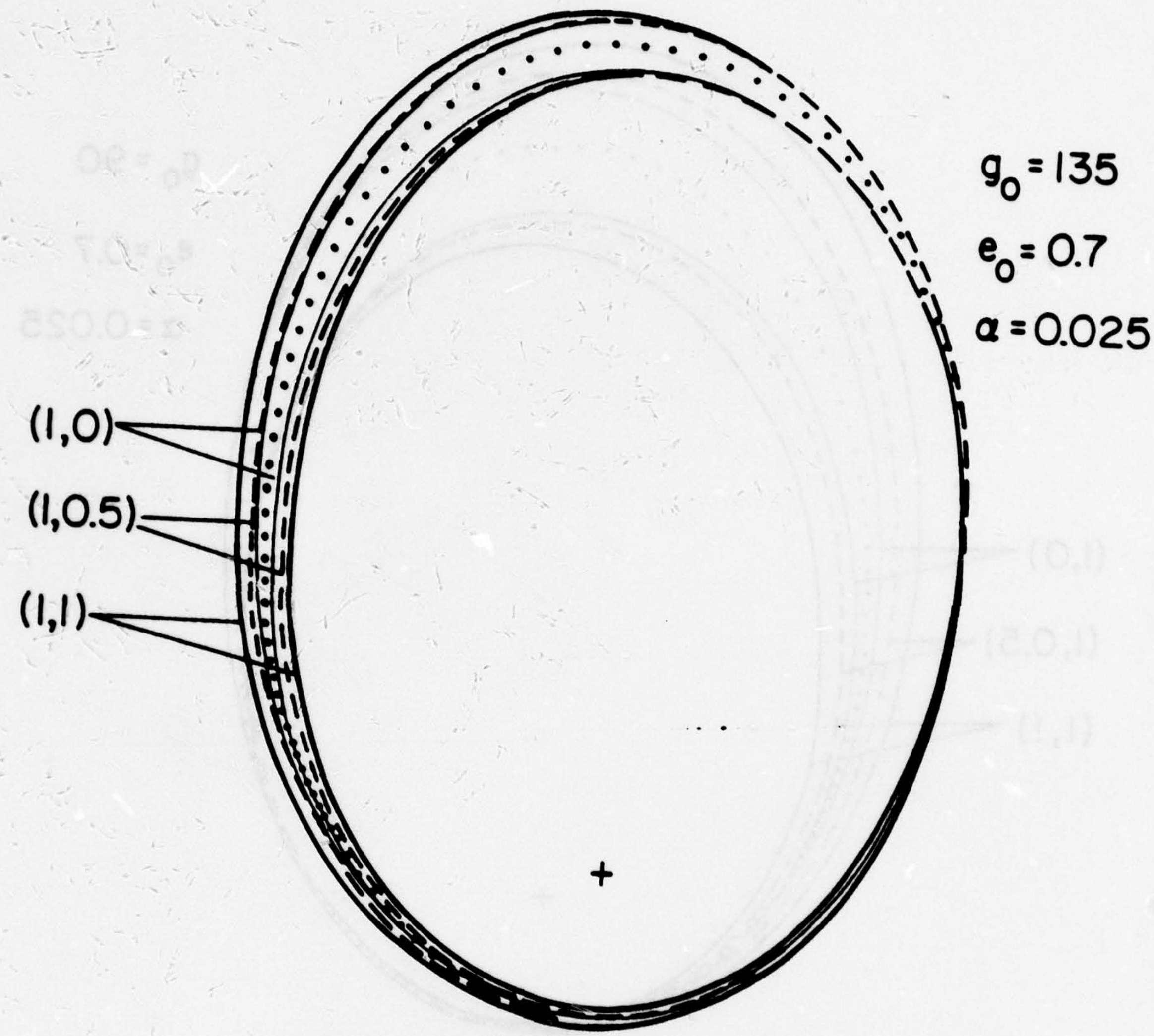

Fig. 4c - Same as 4a but $g_o = 135^o$

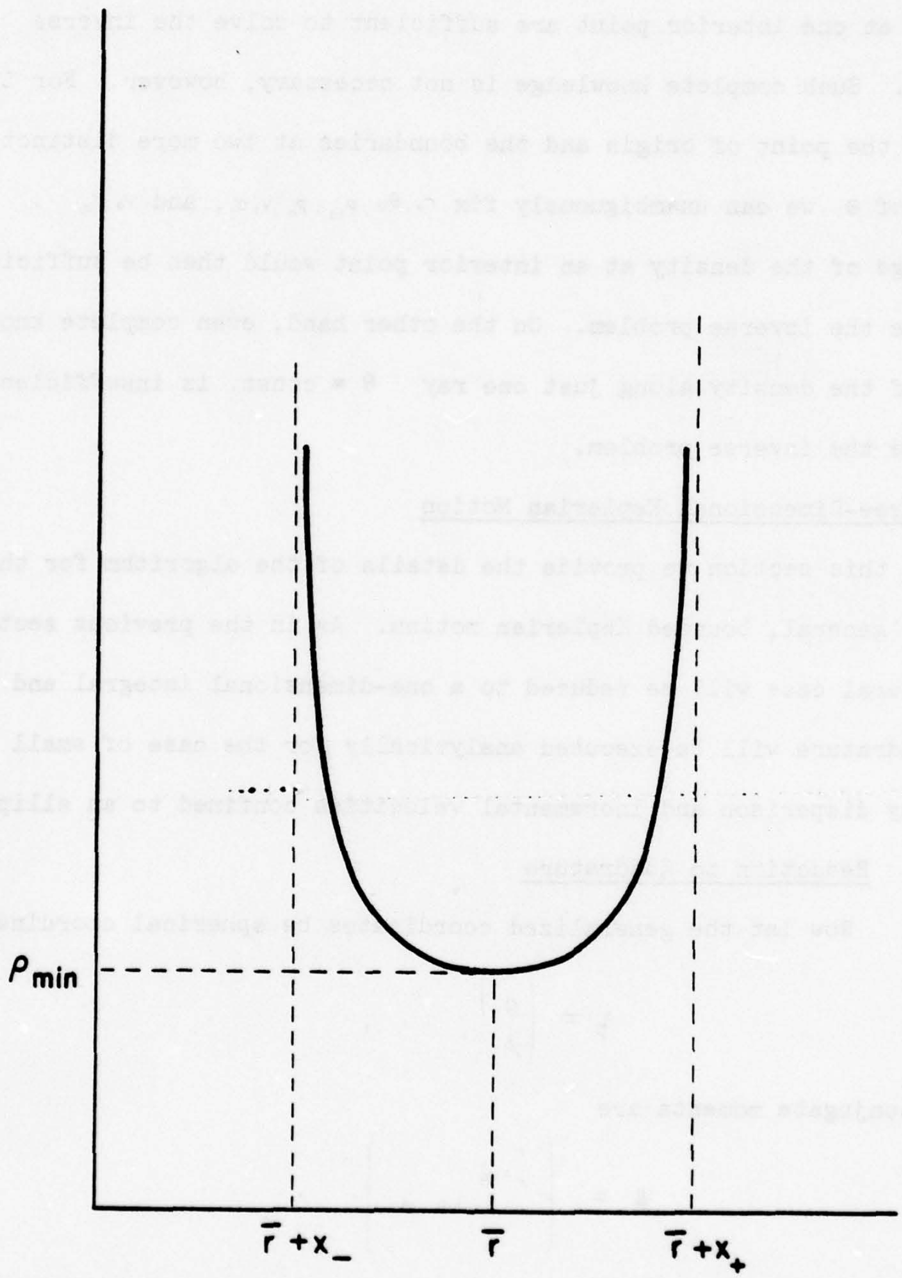

Fig. 5 - The radial dependence of Keplerian spatial density in the orbit plane of the parent body for the momentum distribution function given by equation (60)

interior point. The central density at any point for which $\theta \neq \theta_0$ would suffice. We conclude that knowledge of the boundary and the density at one interior point are sufficient to solve the inverse problem. Such complete knowledge is not necessary, however. For if we know the point of origin and the boundaries at two more distinct values of θ we can unambiguously fix $r_0, \theta_0, p_{10}, p_0, v_0\sigma_1$, and $v_0\sigma_2$. Knowledge of the density at an interior point would then be sufficient to solve the inverse problem. On the other hand, even complete knowledge of the density along just one ray θ = const. is insufficient to solve the inverse problem.

Three-Dimensional Keplerian Motion

In this section we provide the details of the algorithm for the case of general, bounded Keplerian motion. As in the previous section, the general case will be reduced to a one-dimensional integral and then the quadrature will be executed analytically for the case of small velocity disperison and incremental velocities confined to an ellipsoid.

A. Reduction to Quadrature

Now let the generalized coordinates be spherical coordinates

$$q = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} ,$$

whose conjugate momenta are

$$p = \begin{pmatrix} \dot{r} \\ r^2 \dot{\theta} \\ r^2 \cos^2\theta \, \dot{\phi} \end{pmatrix} .$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2} (p_1^2 + p_2^2/r^2 + p_3^2/r^2 \cos^2\theta) - \mu/r .$$

Let the coordinate system be oriented such that the parent orbit lies in the plane $\theta = 0$ and the initial point is

$$q_0 = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix} .$$

The Delaunay variables are again a convenient linear combination of the action and angle variables so we have

$$\underset{\sim}{w} = \begin{pmatrix} \ell \\ g \\ h \end{pmatrix} \quad , \quad \underset{\sim}{j} = \begin{pmatrix} L \\ G \\ H \end{pmatrix} .$$

The partition to account for the degeneracy of the system is

$$\underset{\sim}{w}' = \begin{pmatrix} g \\ h \end{pmatrix} \quad , \quad \underset{\sim}{v}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$w'' = \ell \quad , \quad v'' = \mu^2 L^{-3} .$$

The well-known relations connecting (q, p) and $(\underset{\sim}{w}, \underset{\sim}{j})$ are

$$L = \mu \left(2\mu/r - p_1^2 - p_2^2/r^2 - p_3^2/r^2 \cos^2\theta \right)^{-1/2}$$

$$G = \left(p_2^2 + p_3^2/\cos^2\theta \right)^{1/2}$$

$$H = p_3 \tag{73}$$

$$r = G^2/\mu \left[1 + e \cos(u - g) \right]$$

$$\sin\theta = \sin i \ \sin u$$

and $\quad \phi = h + \psi$

where $\quad e^2 = 1 - G^2/L^2 \ , \ \cos i = H/G \ , \ \sin\psi = \tan\theta / \tan i \ ,$

and $u = g + v(e, \ell)$ where $v(e, \ell)$ is a periodic function of ℓ .

The initial phase-space distribution function is

$$\varphi_0 = \mathcal{F}(p_1, p_2, p_3)\, \delta(r - r_0)\, \delta(\theta)\, \delta(\phi) / r_0^2 \quad .$$

Using the relations (73) and equation (50), one obtains the following limit of the phase-space distribution function

$$\varphi(r, \theta, \phi, p_1, p_2, p_3) = \mathcal{F}(\Psi_1, \Psi_2, \Psi_3)\, \delta(\eta)\, \delta(\zeta) / |J_1|\, r_0^2 \tag{74}$$

where

$$\Psi_1(p_1, p_2, p_3) = \left[p_1^2 + (p_2^2 + p_3^2/\cos^2\theta)(1/r^2 - 1/r_0^2) + 2\mu\,(1/r_0 - 1/r) \right]^{1/2} ;$$

$$\Psi_2(p_2, p_3) = (p_2^2 + p_3^2 \tan^2\theta)^{1/2} \quad ,$$

$$\eta = \arcsin\left[\left(1 - \frac{H^2}{G^2}\right)^{-1/2} \sin\theta \right] - \arcsin\left[\left(1 - \frac{G^2}{H^2}\right)^{-1/2} \left(\frac{G^2}{r} - 1\right) \right] - \arccos\left[\left(1 - \frac{G^2}{H^2}\right)^{-1/2} \left(\frac{G^2}{r_0} - 1\right) \right]$$

and

$$\zeta = \phi - \arcsin\left[\left(\frac{G^2}{H^2} - 1\right)^{-1/2} \tan\theta \right] \quad .$$

In the last two equations, the variables L, G, H must be regarded as functions of p_1, p_2, p_3, r and θ as given in equations (73). In deriving these expressions, use has been made of the fact that h and the initial value of u are either 0 or π because of the choice of the coordinate system. The Jacobian determinant appearing in equation (74) is

$$J_1 = \left. \frac{\partial(r, \theta, \phi)}{\partial(\ell, g, h)} \right|_{j, w_0}$$

$$= \mu\,(2\mu/r - p_1^2 - p_2^2/r^2 - p_3^2/r^2\cos^2\theta)^{-3/2}\, \Psi_1 (p_2^2 \cos^2\theta + p_3^2 \sin^2\theta)^{1/2} / \Psi_2$$

The limiting spatial density function is now

$$\bar{\rho} = \int \bar{\varphi} \; dp_1 \, dp_2 \, dp_3 \; . \tag{75}$$

To evaluate the integral (75) it is convenient to transform coordinates from p_1, p_2, p_3 to η, ζ, p_3 to obtain

$$\bar{\rho} = \int \mathcal{F}\left\{\psi_1\left[p_1(0,0,p_3), p_2(0,0,p_3), p_3\right], \psi_2\left[p_2(0,0,p_3), p_3\right]\right\} \times \left|\frac{\partial(p_1, p_2)}{\partial(\eta, \zeta)}\right| \frac{dp_3}{|J_1|\, r_o^2} \tag{76}$$

Omitting the details of the derivation, the final expressions for the first two arguments of $\mathcal{F}$ in equation (76) are

$$\chi_1^2(p_3) \equiv \psi_1^2\left[p_1(0,0,p_3), p_2(0,0,p_3), p_3\right]$$

$$= \frac{\cos^2\theta \; \mu^2}{p_3^2(1+\sin^2\theta \cot^2\phi)} \; \frac{(\beta - \beta_o \cos\phi \cos\theta)^2}{1-\cos^2\theta \cos^2\phi} \, , \tag{77}$$

and $$\chi_2^2(p_3) \equiv \psi_2^2\left[p_2(0,0,p_3), p_3\right]$$

$$= p_3^2 \tan^2\theta / \sin^2\phi \, , \tag{78}$$

where $\beta = (p_3^2/r\mu)(\sec^2\theta + \tan^2\theta/\tan^2\phi) - 1$

and

$$\beta_0 = (p_3^2/r_0\mu)(\sec^2\theta + \tan^2\theta/\tan^2\phi) - 1 \qquad .$$

For the second Jacobian determinant in equation (76), one obtains

$$\frac{\partial(p_1, p_2)}{\partial(\eta, \zeta)} = \left| \frac{\mu \sin\theta\,(\beta - \beta_0 \cos\theta \cos\phi)}{(1+\sin^2\theta \cot^2\phi)^{1/2}\,(1-\cos^2\theta\cos^2\phi)\sin^2\phi} \right| \tag{79}$$

where it is understood that the Jacobian has been evaluated for $\eta = \zeta = 0$.

Assembling the above results, we obtain the following one-dimensional integral for $\bar{\rho}$:

$$\bar{\rho} = \int \mathcal{G}\left[\frac{(\beta-\beta_0\cos\theta\cos\phi)\mu}{\sqrt{AB}\;p_3},\ \frac{p_3\tan\theta}{\sin\phi},\ p_3 \right] \frac{\mu^2 \left| B - \beta^2 - \beta_0^2 + 2\beta\beta_0 \cos\theta\cos\phi \right|^{3/2}}{p_3^2 \cos\theta \sin^2\phi\,(AB)^{3/2}} \frac{dp_3}{r_0^2} \tag{80}$$

where the abbreviations

$$A = (1+\sin^2\theta\cot^2\phi)/\cos^2\theta$$

$$B = 1 - \cos^2\theta\cos^2\phi$$

have been introduced.

Equation (80) reduces the calculation of $\bar{\rho}(r,\theta,\phi)$ for arbitrary $\mathcal{G}(p_1,p_2,p_3)$, to a single quadrature. As in the preceeding section we next consider an important special case for which the integral (80) can be evaluated analytically.

B. Small Initial Velocity Increments Concentrated on an Ellipsoid

Let us now consider the case where the momentum distribution function is

$$\mathfrak{f}(p_1, p_2, p_3) = \delta\left[\frac{(p_1 - p_{10})^2}{\sigma_1^2} + \frac{p_2^2}{r_0^2 \sigma_2^2} + \frac{(p_3 - p_0)^2}{r_0^2 \sigma_3^2} - v_0^2\right] \frac{1}{\pi \sigma_1 \sigma_2 \sigma_3 r_0^2 v_0^3} \qquad (81)$$

This describes anisotropic dispersion from a parent orbit whose initial conditions are $r_0, 0, 0; p_{10}, 0, p_0$. The anisotropy is of the special form that the momentum increments lie on a triaxial ellipsoid. Were $\sigma_1 = \sigma_2 = \sigma_3$, the dispersion would be isotropic and the incremental velocity of each particle would be $\sigma_1 v_0$. Were $p_{10} = 0$ and $p_0^2/\mu = r_0$, the parent orbit would be circular.

The integral (59) becomes

$$\bar{\rho} = \int \delta\left[\xi(p_3)\right] \frac{\mu^2 \left| B - \beta_0^2 - \beta^2 + 2\beta\beta_0 \cos\theta \cos\phi \right|^{3/2}}{p_3^2 \cos\theta \sin^2\phi \, (AB)^{3/2} (\pi \sigma_1 \sigma_2 \sigma_3 r_0^4 v_0^3)} \, dp_3 \quad , \qquad (82)$$

where

$$\xi(p_3) = \frac{1}{AB\sigma_1^2}\left[\frac{(\beta - \beta_0 \cos\theta \cos\phi)\mu}{p_3} - \sqrt{AB}\, p_{10}\right]^2 + \left[\frac{p_3 \tan\theta}{r_0 \sigma_2 \sin\phi}\right]^2 + \frac{(p_3 - p_0)^2}{r_0^2 \sigma_3^2} - v_0^2 \quad .$$

Now, we again exploit the fact that the incremental velocities are expected to be small relative to the velocity of the parent body, and write

$$r = (p_0^2/\mu)(\bar{r} + x)$$

$$p_3^* = p_0(1 + \epsilon)$$

where $\bar{r}$ is given by equation (64); $|x|, |\epsilon|$ and $|\theta| \ll 1$; and $\xi(p_3^*) = 0$.

It follows that $A = 1 + O(\theta^2)$ and $B = \sin^2\phi + O(\theta^2)$. After retaining the lowest order terms in x and θ in the evaluation of the integral (82) one obtains

$$\bar{\rho}(x,\theta,\phi) = \frac{\mu \bar{r}^2 (1-e_o^2)^{3/2}}{\pi \sigma_1 \sigma_2 \sigma_3 r_o^3 v_o^3 p_o} \quad \frac{\Delta^{-1/2}}{s} \tag{83}$$

where $s = \sin\phi \quad , \quad c = \cos\phi$

and

$$\Delta = \left\{ \left(\frac{p_o^2}{\mu r_o}\right)^2 \frac{s^2}{\sigma_3^2} + \frac{[2(1-c) - (p_o p_{10}/\mu)s]^2}{\sigma_1^2} \right\} \left[\left(\frac{r_o v_o}{p_o}\right)^2 - \frac{\theta^2}{s^2 \sigma_2^2} \right] - \frac{x^2}{\sigma_1^2 \sigma_3^2 \bar{r}^4} \; . \tag{84}$$

The asymptotic domain is bounded by the surface defined by $\Delta = 0$, that is, the surface

$$\frac{(\mu r_o/p_o^2 - \bar{r})^2}{\bar{r}^4 \left\{ \left(\frac{p_o^2}{\mu r_o}\right) s^2 \sigma_1^2 + [2(1-c) - (p_o p_{10}/\mu)s]^2 \sigma_2^2 \right\}} + \frac{(\bar{r}\theta)^2}{(\bar{r} s \sigma_2)^2} = \left(\frac{r_o v_o}{p_o}\right)^2 \tag{85}$$

C. <u>Discussion of the Three-Dimensional Case</u>

Following the procedure of the previous section, it is easily shown that equation (84) reduces to that obtained by Kotsakis in the case of isotropic dispersion from a circular orbit. Similarly, if

one calculates

$$\bar{\bar{\rho}}(\phi) = \frac{\displaystyle\int_{-b}^{b}\int_{-a\sqrt{1-\theta^2/b^2}}^{a\sqrt{1-\theta^2/b^2}} \bar{\rho}(x,\theta,\phi)\, dx\, d\theta}{\displaystyle\pi r_0^3 ab \int_0^{2\pi}\int_{-b}^{b}\int_{-a\sqrt{1-\theta^2/b^2}}^{a\sqrt{1-\theta^2/b^2}} \bar{\rho}\, dx\, d\theta\, d\phi} ,$$

where a and b are the semi-axes of the cross-sections of the domain by planes $\phi =$ const., one agrees with the averaged density reported by Kotsakis except for a factor of two. This discrepancy arises because of an apparent slip, by a factor of two, in Kotsakis' calculation of the cross-sectional area.

The cross-sections of the asymptotic domain by a plane $\phi =$ const. remain elliptical as in the isotropic, circular parent orbit case. The semi-axes of these ellipses become

$$\text{x axis: } \left(\frac{r_0 v_0}{p_0}\right)\bar{r}^2\left(\frac{p_0^2}{\mu}\right)\left\{\left(\frac{p_0^2}{\mu r_0}\right)^2 s^2\sigma_1^2 + \left[2(1-c) - (p_0 p_{10}/\mu)\, s\right]^2 \sigma_3^2\right\} ,$$

$$\text{y axis: } \left(\frac{r_0 v_0}{p_0}\right)\left(\frac{p_0^2}{\mu}\right)\bar{r}\, s\, \sigma_2 .$$

A series of these ellipses is illustrated in Figure (6) for the case $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $e_0 = 0.7$, $g_0 = 135°$.

In sections of the bounding volume by the planes ϕ const., the iso-density contours are ellipses. The density is again singular at the boundary of the domain. The density minimum is attained on the

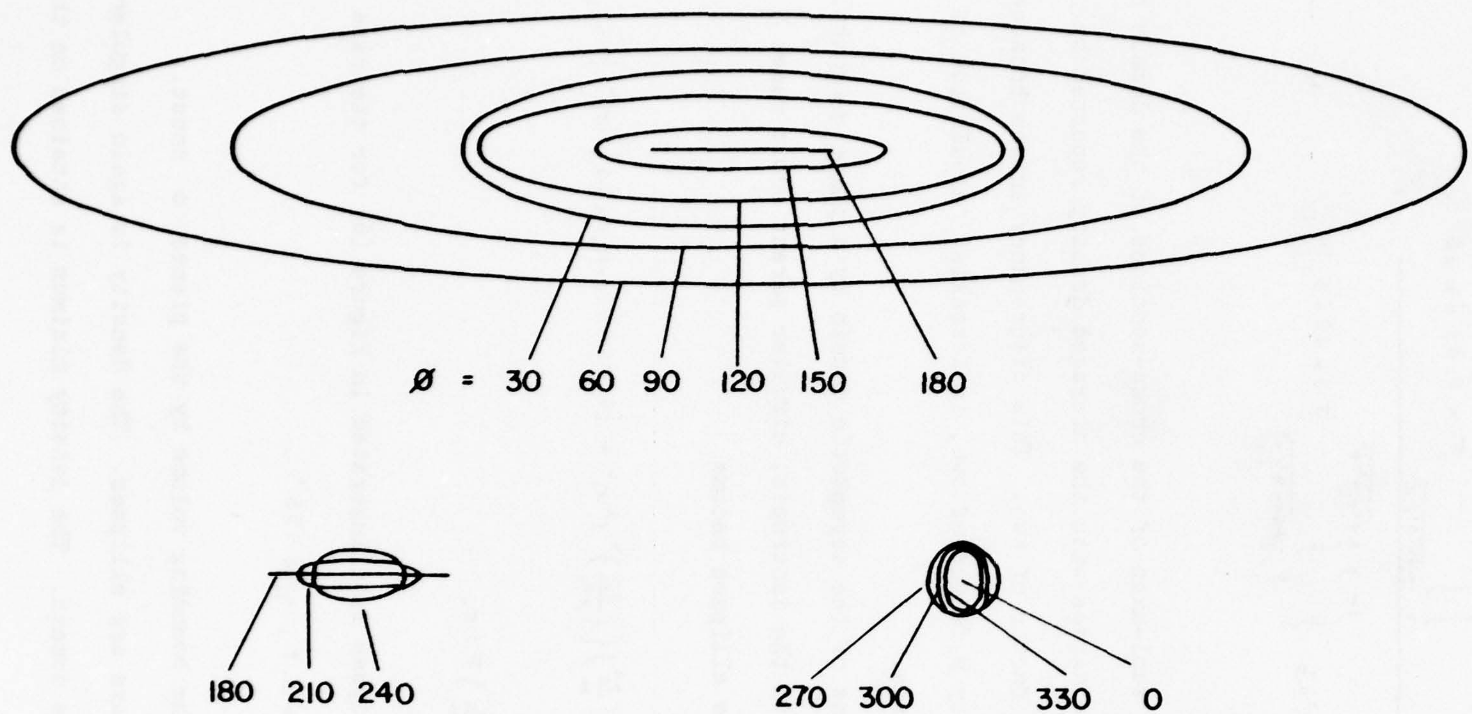

Fig. 6 - Sections of the Keplerian asymptotic domain by planes normal to the orbit plane of the parent. For the case illustrated, $e_o = 0.7$ and $g_o = 135^0$

parent orbit.

The cross-section of the bounding domain by the plane $\theta = 0$ is the bounding curves described in the previous section. To transfer the two-dimensional results to the three-dimensional case one must replace σ_2 by σ_3 .

Particles Orbiting an Oblate Primary

The results obtained in the previous section are unrealistic in the sense that perturbations due to the oblateness of the primary, or the attraction of a third body, would precess the orbits and destroy the highly stylized structure of the Keplerian case. One might argue that when the perturbations are small there will be an intermediate period of time when the Keplerian dispersion will dominate. During this intermediate period, the structure we have just discussed would provide a decent approximation to the state of the ensemble. Nevertheless, in the limit $t \to \infty$ the perturbations would eventually dominate.

Fortunately, the theory developed in Section 3 is sufficiently general to deal with the effect of perturbations. In this section we shall use the algorithm to determine the boundary of the asymptotic domain when the particles orbit an oblate primary. The discussion of the density in this case will be left for a future paper. In order to apply the theory, it is necessary to deal with systems governed by a separable Hamiltonian. The Hamiltonian describing motion about an oblate primary is not separable, except in the case of the Vinti potential. However, the motion of a particle about

an oblate primary may be described accurately by an intermediary orbit which is based on a separable Hamiltonian (Garfinkel, 1959, Aksnes, 1972).

A. Boundary of the Asymptotic Domain

Let us again choose spherical coordinates to describe the motion of the particles. Let the plane $\theta = 0$ be the equatorial plane of the oblate primary. Let the point $(r_0, \theta_0, 0)$ be the initial point.

The Hamiltonian for a spherical coordinate intermediary orbit is (Aksnes, 1972)

$$\mathcal{H} = \frac{1}{2}\left(p_1^2 + p_2^2/r^2 + p_3^2/r^2\cos^2\theta\right) - \mu/r + \mu c_1 J_2\left(\frac{a_e}{r}\right)^2 P_2(\sin\theta) \ , \tag{86}$$

where J_2 is the second zonal geopotential coefficient ($\sim 10^{-3}$ for the earth), a_e is the equatorial radius of the primary, P_2 is the second Legendre polynomial, and c_1 is a constant which may depend on the momenta. Associated with this Hamiltonian are Delaunay type variables which we shall use as a convenient combination of action and angle variables. The relations connecting these generalized Delaunay variables to the coordinates and momenta are more complicated than equations (73). For our purposes we need list only the relations

$$L = \mu\left(2\mu/r - p_1^2 - G^2/r^2\right)^{-1/2} \ ,$$

$$G^4 - \left(p_2^2 + p_3^2/\cos^2\theta\right) G^2 - 2\mu^2 J_2 a_e^2 P_2(\sin\theta) = 0 \ ,$$

$$H = p_3 \ . \tag{87}$$

The importance of the Hamiltonian (86), in the present application, is that it is separable, that the frequencies associated with L, G, H are non-zero and linearly independent almost everywhere, and that it accurately describes the motion of a particle about an oblate primary.

Since the dynamical system is non-degenerate, it is not necessary to partition the angle variables, and the integral (51) gives the spatial density. This integral becomes

$$\bar{\rho}(r,\theta) = \int \mathcal{F}(\Psi_1, \Psi_2, p_3)\, dp_1\, dp_2\, dp_3 \,/\, |J_1(r,\theta, p_1, p_2, p_3)| \tag{88}$$

where

$$\Psi_1^2 = p_1^2 + G^2(1/r^2 - 1/r_0^2) + 2\mu(1/r_0 - 1/r) \quad ,$$

$$\Psi_2^2 = G^2 - p_3^2/\cos^2\theta_0 - 2\mu c_1 J_2 P_2(\sin\theta_0)$$

and

$$G = G(\theta, p_2, p_3) \ .$$

We see immediately that coordinate ϕ is absent in equation (88) so that the density and the bounding surface are rotationally symmetric about the axis of symmetry of the primary.

As long as J_2 is not zero, no matter how small, we must deal with the three-dimensional integral (88) instead of the one-dimensional integral derived for the degenerate, Keplerian case. Having accounted for this, however, we may now neglect the effects of J_2 and still obtain an accurate description of the state of the ensemble. A

consequence of this is that the relations (87) may be replaced by the relations (73) in what follows.

To discuss the boundary of the asymptotic domain, we shall treat the following form of $\mathcal{F}(\underline{p})$

$$\mathcal{F}(p_1, p_2, p_3) = \delta\left[p_1^2 + (p_2 - p_{20})^2/r_0^2 + (p_3 - p_{30})^2/r_0^2\cos^2\theta_0 - V_0^2 \right] \tag{89}$$

where $r_0, \theta_0, 0, 0, p_{20}, p_{30}$ are the initial conditions for a circular parent orbit. The current choice of coordinates does not allow us to assume that $p_{20} = 0$ as in equation (81). From equation (89) it follows that

$$\mathcal{F}(\psi_1, \psi_2, p_3) = \delta\left[\xi(p_1, p_2, p_3) \right] , \tag{90}$$

where

$$\begin{aligned} \xi = p_1^2 + (\psi_2 - p_{20} r^2/r_0^2)^2/r^2 + (p_3 - p_{30} r^2/r_0^2)^2/r^2\cos^2\theta \\ - (\mu/r_0)(\alpha^2 + 2r_0/r + r^2/r_0^2 - 3) \end{aligned} , \tag{91}$$

$$\psi_2^2 = p_2^2 + p_3^2(\sec^2\theta - \sec^2\theta_0) , \tag{92}$$

and

$$\alpha^2 = (V_0/V_c)^2 = r_0 V_0^2/\mu .$$

In equation (91), use has been made of the relations

$$p_{20}^2 = \mu r_0 (1 - \cos^2 i/\cos^2\theta_0)$$

$$p_{30}^2 = \mu r_0 \cos^2 i ,$$

where i is the inclination of the parent orbit.

A point (r, θ, ϕ) lies in the asymptotic domain of the ensemble if the equation

$$\xi(r, \theta, p_1, p_2, p_3) = 0 \tag{93}$$

has a solution. This equation alone, however, is insufficient to determine the domain. We must also satisfy the constraints

$$\psi_2^2 - p_3^2(\sec^2\theta - \sec^2\theta_0) > 0 \tag{94}$$

and

$$p_1^2 + (\psi_2^2 + p_3^2 \sec^2\theta_0)(1/r^2 - 1/r_0^2) + 2\mu(1/r_0 - 1/r) > 0 \,. \tag{95}$$

Now let us concentrate on the case of small velocity increments. This allows us to represent

$$r = r_0(1 + x)$$

and to assume

$$\alpha \sim O(x) \,, \quad |x| \ll 1 \quad .$$

Further, let us choose units such that $\mu = 1$, $r_0 = 1$. With these simplifications, equations (93) - (95) become

$$\frac{p_1^2}{\alpha^2 + 3x^2} + \frac{[\psi_2 - p_{20}(1+x)^2]^2}{\alpha^2 + 3x^2} + \frac{[p_3 - p_{30}(1+x)^2]^2}{(\alpha^2 + 3x^2)\cos^2\theta_0} = 1 \tag{96}$$

$$p_1^2 + (\psi_2^2 + p_3^2 \sec^2\theta_0)(3x^2 - 2x) > 2(x^2 - x) \tag{97}$$

$$\psi_2^2 > p_3^2 (\sec^2\theta - \sec^2\theta_0) \quad . \tag{98}$$

Equations (96) - (98) provide a useful geometrical interpretation of the problem of locating the boundaries of the asymptotic domain. Regard equations (96) - (98) in coordinates p_1, ψ_2, p_3 . Equation (96) represents an oblate spheroid centered at $[\,0,\ p_{20}(1+x)^2,\ p_{30}(1+x)^2\,]$. The boundary of the region for which equation (97) is satisfied is either a triaxial ellipsoid ($x < 0$) or an elliptical hyperboloid of one sheet ($x > 0$), both centered at the origin. If $\theta < \theta_0$, equation (98) is always satisfied. If $\theta > \theta_0$, the boundary of the region for which equation (98) is satisfied is a wedge. Thus, our problem is to find those pairs (x, θ) for which the ellipsoid (96) intersects or lies within the wedge (98) and the ellipsoid/hyperboloid (97).

Let

$$\epsilon_* = \alpha \sqrt{\cos^2\theta_0 - \cos^2 i} \;/ \sin i \tag{99}$$

When $|\theta| < i$ the equation and the constraints are satisfied if

$$x^2 < 16\alpha^2 \tag{100}$$

This part of the asymptotic domain is independent of ϑ and is the volume between the spheres $r = 1 \pm 4\alpha$, but outside the cones $\theta = \pm i$.

When $i < |\theta| < i + \epsilon_*$, the equation and the constraints are satisfied if

$$x^2 < x_*^2 \tag{101}$$

where

$$\theta = i + \epsilon$$

and

$$x_*^2 = 16\alpha^2 \left[1 - (\epsilon/\epsilon_*)^2 \right] \quad . \tag{102}$$

The asymptotic domain, as described by equations (100) and (101), is illustrated in Figure 7. This figure shows the cross-section of the domain by a plane ϕ = const. The domain is the volume of revolution swept out as the curve is rotated about the axis of symmetry of the primary. An important feature of the bounding curve is that it depends on the initial latitude. The perturbations do not destroy the record of the initial point. In the Keplerian case, the asymptotic domain is bounded by a toroid which is aligned with the plane of the parent orbit but is constricted to a point at the place of origin and pinched to a line 180° away in true anomaly. When oblateness perturbations are considered, the asymptotic domain is a toroid whose axis is the axis of symmetry of the oblate primary.

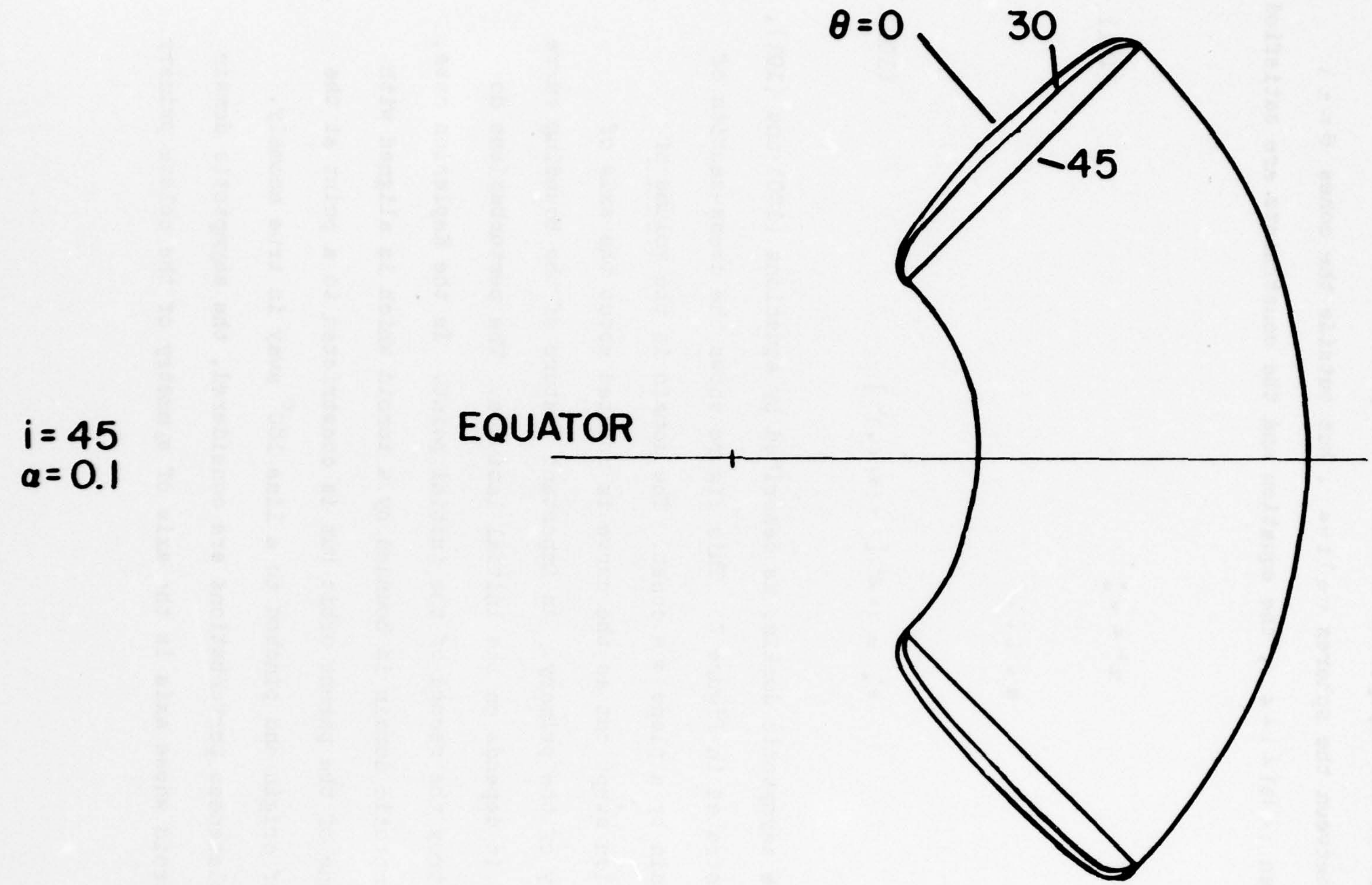

Fig. 7 - Cross section of the boundary of the asymptotic domain for particles orbiting an oblate primary. Curves are shown for initial latitudes of 0^o, 30^o, and 45^o

THE INVERSE PROBLEM

There is a large class of problems in the physical sciences, generally known as inverse problems, which are concerned with the determination of models of physical systems from indirect, and usually incomplete, observational data. These problems abound in geophysics (Parker, 1977; Gilbert, 1971) and one may cite as examples the problems of determining the density of the earth's interior from seismic data, the structure of the ionosphere from radio wave propagation characteristics, and the structure of the geomagnetic field from satellite-born magnetometer measurements. Excellent expositions of the mathematical aspects of these problems have been given by Keller (1976), Prosser (1977), and Backus (1971).

In a sense, the problem of orbit determination is an inverse problem. Certainly there are similarities between the techniques used by Backus and Gilbert to solve inverse problems and the differential correction schemes used to estimate orbital elements from observations of the position of a celestial body. However, there is also a fundamental difference between estimation problems and inverse problems. The former involve the estimation of a finite number of parameters from an over determined data set while the latter seeks a continuous model from a woefully inadequate data set. It is useful to draw the analogy between the relation of maximum-minima problems of elementary calculus to problems of the calculus of variations, one one hand, and estimation problems to inverse problems on the other. A good example of an inverse problem in modern day celestial mechanics is the

determination of atmospheric density models from the study of decaying artificial earth satellites.

In this section we consider the inverse problem associated with an ensemble of particles which emanate simultaneously from a point source with a continuous distribution of momenta. The problem is to determine the initial momentum distribution from observations of the evolving ensemble.

The inverse problem for the particle ensemble system, or any system for that matter, is completely posed only when one has specified precisely what observations are to be used. For an ensemble of particles, knowledge of the time and position of initial dispersion and a complete knowledge of the spatial distribution of particles at a later time admit a relatively simple solution of the problem. In practice, however, one seldom has such complete knowledge of the system. Usually, only a sample of the ensemble is observed and the time and place of initial dispersion may not be known. This occurs, for example, in the cases of fragmentation of artificial earth satellites and the formation and evolution of meteor streams.

Thus, the study of our inverse problem must be based on properties which can be estimated from a subset of the ensemble. The theory developed in the sequel is based on the moments of particle position relative to a body in the orbit of the parent. A truncation at some order is inevitable, and the present theory is truncated at the fourth order moments. In statistical terminology, the moments of third order incorporate <u>skewness</u> of the distribution while fourth order moments

incorporate kurtosis.

Our mathematical treatment of the inverse problem will proceed in stages from the ideal situation of complete knowledge of $\rho(q)$, t_o , and place of breakup to cases of progressively less observational material. We shall deal only with linear dynamical systems in this section. Further, let us consider only conservative systems so that $\Gamma(q,p,t)=0$ [Eq. (7)]. This is done just for convenience and there is no difficulty in modifying the theory for non-zero Γ .

Given $\rho(q)$ and Time and Place of Initial Disintegration

To begin the study of the inverse problem, let us examine the case in which one knows the spatial density completely as well as the time of initial dispersion, an equivalent statement is that the time t is known. From equation (21), the spatial density is related to the momentum distribution function according to

$$\rho(q,t) = \frac{1}{|\det V(0,t)|} G\left[M(0,t)\,q - Y(0,t)\,V^{-1}(0,t)\,q_*\right] \tag{21 bis}$$

We seek an expression for $G(p)$. The simple form of the relation (21) allows us to write immediately

$$G(p) = |\det V(0,t)|\,\rho\left[M^{-1}(0,t)\left(p + Y(0,t)\,V^{-1}(0,t)\,q_*\right)\right] . \tag{103}$$

As an example, suppose $\rho(q)$ were known to be of ellipsoidal form

$$\rho = \mathcal{N} \exp\left\{-\tfrac{1}{2}(\underset{\sim}{q} - \bar{\underset{\sim}{q}}) \cdot A(\underset{\sim}{q} - \bar{\underset{\sim}{q}})\right\} , \tag{104}$$

where A is a symmetric, positive definite matrix and $\bar{\underset{\sim}{q}}$ is a constant. The momentum distribution which produced this particle distribution would be the ellipsoidal distribution

$$G(\underset{\sim}{p}) = \mathcal{N} \,|\det V(0,t)|\, \exp\left\{-\tfrac{1}{2} Q\right\} , \tag{105}$$

with

$$Q = \left[\underset{\sim}{p} + Y(0,t)V^{-1}(0,t)\underset{\sim}{q}_* - M\bar{\underset{\sim}{q}}\right] \cdot \tilde{M}^{-1}(0,t) A M^{-1}(0,t)\left[\underset{\sim}{p} + Y(0,t)V^{-1}(0,t)\underset{\sim}{q}_* - M\bar{\underset{\sim}{q}}\right] .$$

Given Moments of $\rho(\underset{\sim}{q})$ and the Time and Place of Initial Disintegration

Now we relax the requirement on knowledge of $\rho(\underset{\sim}{q})$ and assume knowledge of the low order moments (order zero through four). A solution of the inverse problem may be obtained in this case by introducing expansions for $\rho(\underset{\sim}{q})$ and $G(\underset{\sim}{p})$ in three-dimensional Hermite polynomials and by using equation (103) to relate the known moments to the unknown coefficients in the expansion for $G(\underset{\sim}{p})$. There is, in principle, no necessity to truncate the expansion at fourth order. The development given below may be carried out to any desired order. In practice, however, it seems desirable to truncate the theory at fourth order.

To begin the analysis, let us recall some basic properties of three-dimensional Hermite polynomials (Grad, 1949). The polynomials are defined by

$$\mathcal{H}^{(n)}(\underline{x}) = (-1)^n \omega^{-1} D^n \omega \quad ,$$

with

$$\omega = (2\pi)^{-3/2} \exp\left\{-\tfrac{1}{2}\underline{x}^2\right\} \quad .$$

Here D^n is the differential operator of all partial derivatives of order n and $\mathcal{H}^{(n)}$ is a tensor of order n whose components are polynomials of degree n . Explicit expressions for the polynomials through fourth order are:

$$\mathcal{H}^{(0)} = 1 \quad ,$$

$$\mathcal{H}^{(1)}_i = x_i \quad ,$$

$$\mathcal{H}^{(2)}_{ij} = x_i x_j - \delta_{ij} \quad ,$$

$$\mathcal{H}^{(3)}_{ijk} = x_i x_j x_k - (x_i \delta_{jk} + x_j \delta_{ik} + x_k \delta_{ij}) \quad ,$$

$$\begin{aligned}\mathcal{H}^{(4)}_{ijkl} = x_i x_j x_k x_l &- (x_i x_j \delta_{kl} + x_i x_k \delta_{jl} + x_i x_l \delta_{jk} + x_j x_k \delta_{il} + x_j x_l \delta_{ik} + x_k x_l \delta_{ij}) \\ &+ (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\end{aligned}$$

where δ_{ij} is the Kronecker delta.

If $f(\underline{x})$ is a function satisfying the condition

$$\int_{-\infty}^{\infty} f^2(\underline{x})\, \omega^{-1}\, d\underline{x} < \infty \quad , \qquad (106)$$

then f may be represented by the Hermite expansion

$$f(\underset{\sim}{x}) = \omega\left[a^{(0)} + \sum_{\lambda} a^{(1)}_{\lambda} \mathcal{H}^{(1)}_{\lambda} + \sum_{\mu} a^{(2)}_{\mu} \mathcal{H}^{(2)}_{\mu} + \cdots \right] \tag{107}$$

where the subscripts on $\mathcal{H}^{(n)}$ and $a^{(n)}$ are n-indices. Any function of compact support, hence any spatial density function, satisfies condition (106) and may be represented by the series (107). Note that not all terms in the series (107) are unique because $a^{(n)}_{\lambda}$ occurs once for each permutation of λ. The Hermite coefficients for equation (107) are evaluated from

$$a^{(n)}_{\lambda} = \frac{1}{n!} \int f(\underset{\sim}{x})\, \mathcal{H}^{(n)}_{\lambda}(\underset{\sim}{x})\, d\underset{\sim}{x} \quad . \tag{108}$$

To apply these concepts to the solution of the inverse problem we introduce the variable

$$\underset{\sim}{x} = A\,(\underset{\sim}{p} - \underset{\sim}{p}_0) \quad ,$$

where A is a symmetric, positive definite matrix and $\underset{\sim}{p}_0$ is a constant. A and $\underset{\sim}{p}_0$ are unspecified at this point. Next, represent $G(\underset{\sim}{p})$ by the series

$$G(\underset{\sim}{p}) = \omega(\underset{\sim}{x})\left[a^{(0)} + \sum_{\lambda} a^{(1)}_{\lambda} \mathcal{H}^{(1)}_{\lambda}(\underset{\sim}{x}) + \sum_{\mu} a^{(2)}_{\mu} \mathcal{H}_{\mu}(\underset{\sim}{x}) + \cdots \right] \tag{109}$$

From equation (21), it follows that $\rho(\underset{\sim}{q})$ is represented by the series

$$\rho(\underset{\sim}{q}) = \frac{1}{|\det V|}\, \omega\left[A(M\underset{\sim}{q} - YV^{-1}\underset{\sim}{q}_* - \underset{\sim}{p}_0\right] \left\{ a^{(0)} + \sum_{\lambda} a^{(1)}_{\lambda} \mathcal{H}^{(1)}_{\lambda}\left[A(M\underset{\sim}{q} - YV^{-1}\underset{\sim}{q}_* - \underset{\sim}{p}_0\right] + \cdots \right\} \tag{110}$$

where here, and in the sequal, the time dependent matrices are evaluated at the arguments $(0,t)$.

The next step is to relate the moments of ρ , assumed known, to the unknown coefficients $a_\lambda^{(n)}$.

The zeroth moment, or mean of the density, is

$$\langle \rho \rangle = \int \rho \, d\underset{\sim}{q}$$

$$= \frac{a^{(0)}}{|\det V|} \frac{1}{\det(AM)} \tag{111}$$

The first order moments involve the center of mass of the ensemble and we obtain

$$\langle \rho \rangle \langle \underset{\sim}{q} \rangle = \int \underset{\sim}{q} \rho \, d\underset{\sim}{q}$$

$$= \frac{1}{|\det V| \det(AM)} \left[(AM)^{-1} \underset{\sim}{a}^{(1)} + (M^{-1} Y V^{-1} \underset{\sim}{q}_* + M^{-1} \underset{\sim}{p}_0) a^{(0)} \right]$$

or

$$\langle \underset{\sim}{q} \rangle = \frac{1}{a^{(0)}} \left[(AM)^{-1} \underset{\sim}{a}^{(1)} + (M^{-1} Y V^{-1} \underset{\sim}{q}_* + M^{-1} \underset{\sim}{p}_0) a^{(0)} \right] \tag{112}$$

The second order moments involve the covariances of position relative to the mean. In dyadic notation, we obtain

$$\langle \rho \rangle \langle (\underset{\sim}{q} - \langle \underset{\sim}{q} \rangle)(\underset{\sim}{q} - \langle \underset{\sim}{q} \rangle) \rangle = \int \rho (\underset{\sim}{q} - \langle \underset{\sim}{q} \rangle)(\underset{\sim}{q} - \langle \underset{\sim}{q} \rangle) \, d\underset{\sim}{q}$$

$$= \frac{1}{|\det V| \det(AM)} (AM)^{-1} \left[2 \underset{\approx}{a}^{(2)} + a^{(0)} \underset{\approx}{\delta} - \frac{1}{a^{(0)}} \underset{\sim}{a}^{(1)} \underset{\sim}{a}^{(1)} \right] \tag{113}$$

We now observe that the expansion (109) for $G(\underset{\sim}{p})$ is valid for any positive definite, symmetric matrix A and arbitrary $\underset{\sim}{p}_0$.

Furthermore, it is not possible to separate A and p_o from $\alpha^{(1)}_\lambda$ and $\alpha^{(2)}_\mu$. Therefore, one is free to choose A and p_o and determine $\alpha^{(1)}_\lambda$, $\alpha^{(2)}_\mu$ or vice versa. We shall follow the latter course and choose $\alpha^{(1)}_\lambda = \alpha^{(2)}_\mu = 0$. This simplifies the moment equations considerably without any loss of generality.

Before proceeding, let us ease the notational burden by introducing

$$\underset{\sim}{\varepsilon} = \underset{\sim}{q} - \langle \underset{\sim}{q} \rangle$$

and

$$B = (AM)^{-1} \quad .$$

With these simplifications, equations (112) and (113) become

$$\langle \underset{\sim}{q} \rangle = M^{-1} \left[\underset{\sim}{p}_o + Y V^{-1} \underset{\sim}{q}_* \right] \quad , \tag{114}$$

and

$$\langle \varepsilon \varepsilon \rangle_{ij} = \text{ent}_{ij} \, (\overline{M} A^2 M)^{-1} = \text{ent}_{ij} \, B \overline{B} \quad . \tag{115}$$

The third and fourth order moments are found to be

$$\langle \varepsilon \varepsilon \varepsilon \rangle_{ijk} = [3!/\alpha^{(0)}] \sum_{r,s,t} b_{ir} b_{js} b_{kt} \alpha^{(3)}_{rst} \tag{116}$$

and

$$\langle \varepsilon \varepsilon \varepsilon \varepsilon \rangle_{ijkl} = \sum_{r,s,t,u} b_{ir} b_{js} b_{kt} b_{lu} \left[(4!/\alpha^{(0)}) \alpha^{(4)}_{rstu} + \delta_{rs}\delta_{tu} + \delta_{rt}\delta_{su} + \delta_{ru}\delta_{st} \right] \tag{117}$$

where $b_{ij} = \text{ent}_{ij} \, B$.

To complete the solution, we must solve equations (114)-(117) for $p_0, A, \alpha^{(3)}_{ijk}$ and $\alpha^{(4)}_{ijkl}$. Let

$$\Gamma^{(n)}_{\lambda} = \langle \epsilon\epsilon \cdots \epsilon \rangle_{\lambda}$$

where λ is an n-index. From equations (114) and (115) we obtain

$$p_0 = M\langle q\rangle - YV^{-1}q_* \quad , \tag{118}$$

and

$$A^2 = (M\Gamma^{(2)}\bar{M})^{-1} \quad . \tag{119}$$

Note that these relations are presaged by equations (104) and (105). To invert the third and fourth order relations, introduce the quantities

$$\beta_{ij} = \mathrm{mat}_{ij}\; B^{-1} \quad .$$

Then we obtain

$$\alpha^{(3)}_{ijk} = \frac{\alpha^{(0)}}{3!} \sum \beta_{ir}\beta_{js}\beta_{kt}\,\Gamma^{(3)}_{rst} \tag{120}$$

and

$$\alpha^{(4)}_{ijkl} = \frac{\alpha^{(0)}}{4!}\left[\left(\sum_{r,s,t,u} \beta_{ir}\beta_{js}\beta_{kt}\beta_{lu}\,\Gamma^{(4)}_{rstu}\right) - (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] \tag{121}$$

It is clear from equations (120) and (121) that higher order moment equations would present no inversion difficulties. At all orders, only B^{-1} is needed to execute the inversion.

Assembling the above results, we obtain the following expression

for $G(\underset{\sim}{p})$:

$$G(\underset{\sim}{p}) = \langle \rho \rangle |\det V| \det(AM)\, \omega \left\{ 1 + \frac{1}{3!} \sum_{ijk} \sum_{rst} \beta_{ir} \beta_{js} \beta_{kt} \langle \underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon} \rangle_{rst}\, \mathcal{H}^{(3)}_{ijk} \right.$$

$$\left. + \frac{1}{4!} \sum_{ijkl} \sum_{rstu} \left[\beta_{ir} \beta_{js} \beta_{kt} \beta_{lu} \langle \underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon} \rangle_{rstu} - (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right] \mathcal{H}^{(4)}_{ijkl} + \cdots \right\}$$

where

$$\omega = \exp\left\{ -\tfrac{1}{2} (\underset{\sim}{p} - \underset{\sim}{p}_0) \cdot A^2 (\underset{\sim}{p} - \underset{\sim}{p}_0) \right\} ,$$

$$\underset{\sim}{p}_0 = M \langle \underset{\sim}{q} \rangle - Y V^{-1} \underset{\sim}{q}_* ,$$

and

$$A^2 = (M \Gamma^{(2)} \bar{M})^{-1} .$$

Comments on More General Situations

The case treated in the previous section is usually appropriate for the analysis of satellite breakups. The reason for this is that the most common set of data available on a breakup is a complete set of orbital elements for each fragment and there are several independent methods available for determining the time and place of breakup from such data.

In general, if one knows neither the time nor place of breakup, the inverse problem can be solved if the moments of ρ are known at two different times. The plausibility of this statement follows from the observation that Lambert's theorem provides a method for calculating an orbit given two position measurements and the time interval separating the measurements. A method of solution in this case will be outlined below. But first let us consider a weaker generalization.

Suppose we have two sets of moments associated with known times. Suppose we know neither the time nor place of breakup but do know the orbit of the parent body from which the fragments originated. This last piece of information relates place of breakup to time of breakup.

Thus, we really have to determine only the time of breakup. By applying the algorithm developed in the previous section to both sets of moments one would obtain two momentum distribution functions, $G_1(\mathbf{p},t)$ and $G_2(\mathbf{p},t)$ which depend on the unknown time of breakup. Now form a "loss function"

$$\mathcal{L}(t) = \int [G_1(\mathbf{p},t) - G_2(\mathbf{p},t)]^2 \, d\mathbf{p} \quad . \tag{122}$$

Assuming a unique solution to the inverse problem, one may find the unknown t by finding the zero of $\mathcal{L}(t)$. In practice; because of observational errors, sampling errors, and truncation errors; one should seek the minimum of $\mathcal{L}(t)$ rather than a zero. Once t is found, G_1 and G_2 will agree (or nearly so) and will provide the solution.

The more general case in which we do not assume knowledge of the parent orbit is not conceptionally different. In this case the two sets of moments provide two momentum distribution functions $G_1(\mathbf{p},t,\mathbf{q}_*)$ and $G_2(\mathbf{p},t,\mathbf{q}_*)$ which depend on the unknown time, t, and the unknown position $\mathbf{q}_*$. The appropriate "loss function" is

$$\mathcal{L}(t,\mathbf{q}_*) = \int [G_1(\mathbf{p},t,\mathbf{q}_*) - G_2(\mathbf{p},t,\mathbf{q}_*)]^2 \, d\mathbf{p} \quad . \tag{123}$$

Now, minimizing $\mathcal{L}$ with repsect to the four parameters $\mathbf{q}_*, t$ will yield the solution.

Both of the general cases outlined here would involve considerable numerical effort. It would not be too difficult to write analytical

expressions for the integrals (122) and (123), but that will not be undertaken here. Even with analytic expressions for the integrals, the minimization would involve a non-trivial amount of computation.

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